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Stability investigation of Volterra integral equations by realization theory and frequency-domain methods

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Abstract. Realization theory for linear input-output operators and frequency-domain methods for the solvability of Riccati operator equations are used for the stability and instability investigation of a class of nonlinear Volterra integral equations in some Hilbert space. The key idea is to consider, similar to the Volterra equation, a time-invariant control system generated by an abstract ODE in some weighted Sobolev space, which has the same stability properties as the Volterra equation.

Keywords: infinite dimensional Volterra integral equation, realization theory, absolute instability, frequency-domain methods
AMS subject classification: Primary 45M05, 34G20, secondary 47D06,93B52

0 Introduction

The first step in the derivation of equations describing the dynamic behavior of observations is very often a Volterra integral equation ([1, 20, 28]) which represents causal or input-output properties of such observations or time-series. Stability, oscillating behavior, and other qualitative properties from a Volterra integral equation can be observed directly by frequency-domain methods developed in [9, 22, 27, 31]. However, for other types of dynamic behavior such as instability and dichotomy it is useful to consider together with the given Volterra integral equation an associated realization as evolution equation in some function spaces. Realization theory in Hilbert and Fréchet spaces developed for linear input-output operators, can be found in [13, 14, 21, 25, 33]. First results where linear realization theory was used for the stability investigation of finite-dimensional nonlinear Volterra equations can be found in the papers [2, 6, 7].
In the present paper we continue these investigations for infinite-dimensional Volterra equations. In Section 1 we consider general nonlinear systems in the sense of V. A. Yakubovich ([17, 32]). We show that the use of realization theory ([25]) gives the opportunity to consider abstract time-invariant control systems associated to the processes investigated by V. A. Yakubovich.
The time-invariant control system consists of a weighted Sobolev space, an impulse operator as generator of the shift semigroup, a control operator which is determined by the

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linear part of the kernel of the integral equation, an observation operator and a nonlinear operator which comes from the nonlinear part of the kernel.
In Section 2 we prove the solvability of Riccati operator equations for the realizations of Volterra integral equations in a Hilbert space. This result is very close to a theorem proved by V. A. Brusin ([7]) for the finite-dimensional case. This theorem is used in Section 3 for the derivation of frequency-domain conditions for stability and instability of Volterra integral equations in a Hilbert space.
In Section 4 we investigate a well-known control problem of the flow of gas or liquid in a tube ([8, 15, 19, 23, 27]). The abstract stability theory for Volterra integral equations, developed in Section 3, is used to show that different realizations of the given Volterra equation, such as a PDE with boundary control or an ODE with delay, can have the same stability properties as the Volterra equation.

1 Realization of infinite-dimensional Volterra equations as time invariant control systems in weighted Hilbert spaces

For a Hilbert space Z with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$ the space $L^2_{\rm loc}(\mathbb{R};Z)$ consists of locally L^2-functions on $\mathbb{R}$ with values in Z and with a topology defined by the family of seminorms

$$| z |_n := \Big(\int_{-n}^{n} | z(t) |^2 dt \Big)^{1/2}, \quad n = 1, 2, \ldots .$$

Thus the space $L^2_{\rm loc}(\mathbb{R};Z)$ is considered as a Fréchet space, i.e. as a complete metrizable linear topological space. For any interval $\mathcal{J} \subset \mathbb{R}$ we regard $L^2_{\rm loc}(\mathcal{J};Z)$ as a subspace of $L^2_{\rm loc}(\mathbb{R};Z)$ identifying $L^2_{\rm loc}(\mathcal{J};Z)$ with the set of functions in $L^2_{\rm loc}(\mathbb{R};Z)$ which vanish outside of $\mathcal{J}$. Suppose that Y and U are Hilbert spaces and introduce the Fréchet spaces $L^2_{\rm loc}(\mathbb{R};Y)$ and $L^2_{\rm loc}(\mathbb{R};U)$. Assume that

$$\phi : L^2_{\rm loc}(\mathbb{R}_+;Y) \times \mathbb{R}_+ \times L^2_{\rm loc}(\mathbb{R}_+;Y) \to L^2_{\rm loc}(\mathbb{R}_+;Y) \tag{1.1}$$

is a nonlinear operator generating the *Volterra functional equation*

$$y = \phi\,(y,t,h)\ . \tag{1.2}$$

Assume also that there are a continuous linear operator

$$\mathcal{T} : L^2_{\rm loc}(\mathbb{R};U) \to L^2_{\rm loc}(\mathbb{R};Y) \tag{1.3}$$

and a nonlinear operator

$$\varphi : L^2_{\rm loc}(\mathbb{R}_+;Y) \times \mathbb{R}_+ \to L^2_{\rm loc}(\mathbb{R}_+;U) \tag{1.4}$$

such that the operator (1.1) can be written as

$$\phi(y,t,h) = \mathcal{T}\varphi(y,t) + h(t)\,, \tag{1.5}$$

where $h \in L^2_{\rm loc}(\mathbb{R}_+;Y)$ is considered as *perturbation* or *forcing function.* Thus the Volterra functional equation has the form

$$y = \mathcal{T}u + h\,, \tag{1.6a}$$

$$u = \varphi(y,t)\,. \tag{1.6b}$$

We call (1.6a) the *linear part* and (1.6b) the *nonlinear part* of (1.5). A function $y \in L^2_{\rm loc}(\mathbb{R}_+;Y)$ satisfying (1.6a), (1.6b) for a.a. $t \in \mathbb{R}_+$ is called *solution.* Any pair (y,u), where y is a solution of (1.6a), (1.6.b) and $u = \varphi(y,t)$ is said to be a *process* generated by (1.6a), (1.6b).

The following construction of a "nonlinear dynamical system" is a modified version of [32, 17]. Assume that for each $T \geq 0$ there is a Hermitian form

$$\mathcal{F}_T : L^2(0,T;Y) \times L^2(0,T;U) \to \mathbb{R} \tag{1.7}$$

such that the family of all forms $\{\mathcal{F}_T\}_{T\geq 0}$ is generated by uniformly bounded and self-adjoint linear operators in $L^2(0,T;Y) \times L^2(0,T;U)$. Suppose that for any process (y,u) generated by (1.6a), (1.6b) there exists a sequence $\{T_n\}_{n=1}^{\infty}$ of positive numbers such that $T_n \to +\infty$ and

$$\mathcal{F}_{T_n}(\mathbf{P}_{T_n}y, \widetilde{\mathbf{P}}_{T_n}u) \geq 0,\ n = 1,2,\ldots\,. \tag{1.8}$$

Here $\mathbf{P}_T : L^2_{\rm loc}(\mathbb{R}_+;Y) \to L^2_{\rm loc}(0,T;Y)$ and $\widetilde{\mathbf{P}}_T : L^2_{\rm loc}(\mathbb{R}_+;U) \to L^2_{\rm loc}(0,T;U)$ denote for any $T \geq 0$ the restriction operators on $(0,T)$.

The set of all functions $(y,u) \in L^2_{\rm loc}(\mathbb{R}_+;Y) \times L^2_{\rm loc}(\mathbb{R}_+;U)$ for which (1.8) is satisfied with a fixed sequence $\{T_k\}_{k=1}^{\infty}, T_k \to +\infty$, is denoted by $\mathbf{N}^{\{T_k\}}$. The set of all functions $(y,u) \in L^2_{\rm loc}(\mathbb{R}_+;Y) \times L^2_{\rm loc}(\mathbb{R}_+;U)$ for which there exists at least one sequence $\{T_k\}_{k=1}^{\infty}$ satisfying (1.8) is denoted by $\mathbf{N}$.

Instead of (1.6a), (1.6b) we consider the *extended system*

$$y = \mathcal{T}u + h\,, \tag{1.9a}$$

$$(y,u) \in \mathbf{N}\,. \tag{1.9b}$$

We call (1.9a) the *linear part* and (1.9b) the *nonlinear part* of the extended system. A pair of functions $(y,u) \in L^2_{\rm loc}(\mathbb{R}_+;Y) \times L^2_{\rm loc}(\mathbb{R}_+;U)$ is called a *process* determined by (1.9a), (1.9b) if there exists a function $h \in L^2_{\rm loc}(\mathbb{R}_+;Y)$ such that the triple (y,u,h) satisfies (1.9a), (1.9b) for a.a. $t \geq 0$.

A process (y,u) determined by (1.9a), (1.9b) is said to be *stable* if for any $h \in L^2(\mathbb{R}_+;Y)$ such that (y,u,h) satisfies (1.9a), (1.9b) we have $y \in L^2(\mathbb{R}_+;Y)$ and $u \in L^2(\mathbb{R}_+;U)$. In other case it is called *unstable.*

We say that the extended system (1.9a), (1.9b) is *absolute stable* if there is a $C > 0$ such that for any $h \in L^2(\mathbb{R}_+;Y)$ and any process (y,u), for which (y,u,h) satisfies (1.9a), (1.9b), has the properties $y \in L^2(\mathbb{R}_+;Y), u \in L^2(\mathbb{R}_+;U)$ and

$$|\,y\,|^2_{L^2(\mathbb{R}_+;Y)} + |\,u\,|^2_{L^2(\mathbb{R}_+;U)} \leq C|\,h\,|^2_{L^2(\mathbb{R}_+;Y)}\,. \tag{1.10}$$

The extended system (1.9a), (1.9b) is called *absolute unstable* if for any sequence $\{T_k\}_{k=1}^{\infty}, T_k \to \infty$, there exists a Hermitian operator $M : L^2(\mathbb{R}_+;Y) \to L^2(\mathbb{R}_+;Y)$ such that the set $\mathcal{C} := \{h \,|\, (Mh,h)_{L^2(\mathbb{R}_+;Y)} > 0\}$ is nonempty and for $h \in \mathcal{C}$ we have for any triple (y,u,h) generated by (1.9a), (1.9b) with $\{T_k\}_{k=1}^\infty$ the properties $y \notin L^2(\mathbb{R}_+;Y)$ and $u \notin L^2(\mathbb{R}_+;U)$.

We say ([32]) that the extended system (1.9a), (1.9b) is *minimally stable* if for any $h \in L^2(\mathbb{R}_+;Y)$ and any process (y,u) such that (y,u,h) is generated by (1.9a), (1.9b) with a sequence $\{T_k\}_{k=1}^\infty$ there exists a sequence of processes $\{(y^n,u^n)\}_{n=1}^\infty$ such that for any $n = 1,2,\dots$ the triple (y^n,u^n,h) is generated by (1.9a), (1.9b), $y^n \in L^2(\mathbb{R}_+;Y), u^n \in L^2(\mathbb{R}_+;U)$ and

$$| \, y^n \, |^2_{L^2(\mathbb{R}_+;Y)} + | \, u^n \, |^2_{L^2(\mathbb{R}_+;U)} \geq | \, \mathbf{P}_{T_n} y \, |^2_{L^2(0,T_n;Y)} + | \, \widetilde{\mathbf{P}}_{T_n} u \, |^2_{L^2(0,T_n;U)} \, . \tag{1.11}$$

The extended system (1.9a), (1.9b) is said to be *minimally unstable* ([26]) if there exists a perturbation $h \in L^2(\mathbb{R}_+;Y)$ and a process (y,u) such that (y,u,h) is generated by (1.9a), (1.9b) with the sequence $\{T_k\}_{k=1}^\infty, y \notin L^2(\mathbb{R}_+;Y), u \notin L^2(\mathbb{R}_+;U)$ and there is a sequence of processes $(y^n,u^n)_{n=1}^\infty$ such that the triples (y^n,u^n,h) a generated by (1.9a), (1.9b) and so that $\mathbf{P}_{T_n}(y-y^n) = 0, \widetilde{\mathbf{P}}_{Tn}(u-u^n) = 0, n = 1,2,\dots$.

The following theorem is the specification of two more abstract assertions from [17, 26, 32] to our situation.

Theorem 1.1 *Consider the extended system (1.9a), (1.9b) and the associated forms $\{\mathcal{F}_T\}_{T>0}$ from (1.7). Suppose that there exists a number $\delta > 0$ such that*

$$\begin{aligned} \limsup_{T\to\infty} \mathcal{F}_T(\mathbf{P}_T y, \widetilde{\mathbf{P}}_T u) \leq -\,\delta \left[| \, y \, |^2_{L^2(\mathbb{R}_+;Y)} + | \, u \, |^2_{L^2(\mathbb{R}_+;U)} \right] \\ \forall \, (y,u) \in L^2(\mathbb{R}_+;Y) \times L^2(\mathbb{R}_+;U) : y = \mathcal{T} u \, . \end{aligned} \tag{1.12}$$

Then it holds:

a) *If the system (1.9a), (1.9b) is minimally stable then it is absolute stable.*

b) *If the system (1.9a), (1.9b) is minimally unstable then it is absolute unstable.*

It will be shown in this paper that an abstract result such as Theorem 1.1 can be derived under very similar conditions in a more practical form if we use the realization theory from [13, 21, 25, 33].

Let us discuss for this the realizability of the operator equation (1.9a), (1.9b) as abstract differential equation. Important information comes from the linear operator (1.3) which we call *input-output operator* of the linear part of (1.9a). For any interval $\mathcal{J} \subset \mathbb{R}$, a Hilbert space Z and any $s \in \mathbb{R}$ denote by τ^s the *shift operator* acting on functions $f : \mathcal{J} \to Z$ by

$$\tau^s f(t) := \begin{cases} f(t+s) & \text{if} \quad t+s \in \mathcal{J} \, , \\ 0 & \text{if} \quad t+s \notin \mathcal{J} \, . \end{cases}$$

The input-output operator (1.3) is called *time invariant* if $\tau^t \mathcal{T} = \mathcal{T} \tau^t$ for every $t \in \mathbb{R}$ and is called *causal* if for all $t \geq 0$

$$u(t) = 0 \, , \; \forall \, t \leq T \quad \Rightarrow \quad \mathcal{T} u(t) = 0 \, , \; \forall \, t \leq T \, .$$

This implies that $\mathcal{T}$ in (1.3) is defined by its restriction

$$\mathcal{T} : L^2_{\mathrm{loc}}(\mathbb{R}_+; U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+; Y) \,. \tag{1.13}$$

For any interval $\mathcal{J} \subset \mathbb{R}$, a Hilbert space Z and a parameter $\rho \in \mathbb{R}$ we introduce the weighted spaces $L^2_\rho(\mathcal{J}; Z)$ and $W^{1,2}_\rho(\mathcal{J}; Z)$ by

$$L^2_\rho(\mathcal{J}; Z) := \left\{ f \in L^2_{\mathrm{loc}}(\mathcal{J}; Z) \,\middle|\, \int_{\mathcal{J}} e^{-2\rho t} \,|\, f(t) \,|^2_Z \, dt < \infty \right\}$$

and

$$W^{1,2}_\rho(\mathcal{J}; Z) := \{ f \in L^2_\rho(\mathcal{J}; Z) \,|\, \dot{f} \in L^2_\rho(\mathcal{J}; Z) \} \,.$$

($\dot{f}$ denotes the distribution derivative.) Let us assume that $\mathcal{T}$ from (1.3) can be considered for some $\rho \in \mathbb{R}$ as bounded linear operator

$$\mathcal{T} : L^2_\rho(\mathbb{R}; U) \to L^2_\rho(\mathbb{R}; Y) \,. \tag{1.14}$$

If the property (1.14) is satisfied the input-output operator can be realized as time-invariant control system in weighted Hilbert spaces. The key information for this gives the *Hankel operator* $\mathcal{H}$ associated to the input-output operator $\mathcal{T}$, i.e.

$$\mathcal{H} : L^2_0(\mathbb{R}_-; U) \to L^2(\mathbb{R}_+; Y) \tag{1.15}$$

given by $\mathcal{H} = \mathbf{P}_+ \mathcal{T} \mathbf{P}_-$, where $\mathbf{P}_+ := \mathbf{P}_{\mathbb{R}_+}, \mathbf{P}_- := \mathbf{P}_{\mathbb{R}_-}$ and $\mathbf{P}_E u := \zeta_E u$, where ζ_E is the characteristic function of $E \subset \mathbb{R}$. The space $L^2_0(\mathbb{R}_-; Z)$ is the space of compactly supported square integrable functions which is dual to the space $L^2_{\mathrm{loc}}(\mathbb{R}_+; Z)$ via the pairing $(\psi, \varphi) := \int_{-\infty}^{+\infty} (\psi(-t), \varphi(t))_Z dt$. According to [25] we can describe a state-space description for (1.9a) whose input-output behaviour is given by $\mathcal{T}$ as

$$z(t; z_0, u) = \tau^t z_0 + \tau^t \mathcal{T}(\zeta_{[0,t]} u) \,, \tag{1.16a}$$

$$y(t; z_0, u) = z_0(t) + (\mathcal{T} u)(t) \;,\; t \geq 0 \tag{1.16b}$$

for $z_0 \in Z_0 := L^2_\rho(\mathbb{R}_+; Y)$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; U)$. It is clear that (1.16a), (1.16b) also has the *time invariance property*, i.e.

$$\begin{gathered} z(t+s; z_0, u) = z\left(t; z\left(s; z_0, u\right), \tau^s u\right), \\ y(t+s; z_0, u) = y\left(t; z\left(s; z_0, u\right), \tau^s u\right), \\ \forall\, t, s \in \mathbb{R}_+ ,\; \forall\, z_0 \in Z_0 \,, \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; U) \,. \end{gathered}$$

The state space realization (1.16a), (1.16b) is generated by the time-invariant control evolution system

$$\dot{z} = Az + Bu, \tag{1.17a}$$

$$y = C(z - (\lambda I - A)^{-1} Bu) + \chi(\lambda) u \,, \tag{1.17b}$$

defined in the rigged Hilbert space structure ([4]) (or Gelfand triple [30]) $Z_1 \subset Z_0 \subset Z_{-1}$ with Z_0 as above and $Z_1 := W^{1,2}_\rho(\mathbb{R}_+; Y)$ and the linear operators $A \in \mathcal{L}(Z_1, Z_0) \cap \mathcal{L}(Z_0, Z_{-1}), B \in \mathcal{L}(U, Z_{-1})$ and $C \in \mathcal{L}(Z_1, Y)$ given by

$$\begin{aligned} & A\xi := \dot{\xi} \,,\; \xi \in D(A) \,, \\ & B^* \eta := (\mathcal{H}\eta)(0) \,,\; \eta \in Z^*_{-1} := \{ \eta \in W^{1,2}_\rho(\mathbb{R}_-; Y) \,|\, \eta(0) = 0 \} \,, \\ & Cz := z(0) \,,\; z \in Z_1 \,. \end{aligned}$$

In (1.17b) $\lambda \notin \sigma(A)$ is an arbitrary value. For this values and any other value $\mu \notin \sigma(A)$ the operator $\chi(\lambda) \in \mathcal{L}(U, Y)$ is defined by the identity

$$\chi(\lambda) - \chi(\mu) = (\mu - \lambda) C (\lambda I - A)^{-1} (\mu I - A)^{-1} B \,. \tag{1.18}$$

Note that such an output operator (1.17b) is necessary because the weak solution of (1.17a) will in general not be in Z_1 unless $Bu(t) \in Z_0$, whereas the expression $z - (\mu I - A)^{-1} Bu$ will be in Z_1 whenever $u \in W^{2,2}(0, T; U)$ and $Az(0) + Bu(0) \in Z_0$ ([25]).
If we have the additional properties $B \in \mathcal{L}(U, Z_0)$ or $C \in \mathcal{L}(Z_0, Y)$ the usual transfer operator $C(\lambda I - A)^{-1} B$ makes sense. In this case it follows from (1.18) that $\chi(\lambda) = C(\lambda I - A)^{-1} B$ and (1.17b) goes over in the usual output equation

$$y = Cz \,. \tag{1.19}$$

Instead of the control system (1.17a) defined for a rigged Hilbert space structure we want to consider in the following a time-invariant control system which uses only the pivot space Z_0. Note that many practically important systems with distributed parameters or time delay can be written in this form. Assume for this that the input-output operator $\mathcal{T}$ from (1.14) can be represented as convolution operator

$$(\mathcal{T} u)(t) := \int_0^t K(t - s)\, u\,(s)\, ds \,, \tag{1.20}$$

where $K(\cdot)$ is a certain kernel called ([33]) the *weighting pattern* of $\mathcal{T}$.
Assume that the map $t \in \mathbb{R}_+ \mapsto \mathcal{L}(U, Y)$ is twice piecewise-differentiable and satisfies the following condition: There exists a $\rho_0 > 0$ and a constant $\gamma > 0$ such that

$$\| K(t) \|_{\mathcal{L}(U,Y)} \leq \gamma e^{-\rho_0 t} \,, \quad \forall\, t > 0 \,, \tag{1.21}$$

and

$$\int_0^\infty \left[\| \dot{K}(t) \|^2_{\mathcal{L}(U,Y)} + \| \ddot{K}(t) \|^2_{\mathcal{L}(U,Y)} \right] e^{2\rho_0 t} dt < \infty \,. \tag{1.22}$$

Under these conditions we can choose a state space realization of (1.20) which was used in [7] for the special case $U = Y = \mathbb{R}^n$:

$$Z_0 := W^{1,2}_{-\rho}(0, \infty; Y), \quad \text{where} \quad 0 < \rho < \rho_0 \quad \text{is arbitrary} \,, \tag{1.23}$$

$$D(A) := \left\{ \xi(s) \in W^{1,2}_{-\rho}(0, \infty; Y) \mid \int_0^\infty e^{2\rho s} \| \ddot{\xi}(s) \|^2_Y \, ds < \infty \right\} , \tag{1.24}$$

$$(A\, \xi)(s) := \dot{\xi}(s) \,, \quad \forall\, \xi \in D(A) \,, \tag{1.25}$$

$$(B\eta)(s) := K(s)\eta \,, \quad \forall\, \eta \in U \,, \tag{1.26}$$

$$(Cz)(s) := z(0) \,, \quad \forall\, z \in Z_0 \,. \tag{1.27}$$

Thus we have defined a time-invariant control system

$$\dot{z} = Az + Bu \,, \tag{1.28a}$$

$$y = Cz\,, \tag{1.28b}$$

where A from (1.25) is a closed linear operator that acts in Z_0 given in (1.23) and which has the dense domain of definition $D(A)$ from (1.24). It is clear that A is the generator of some C_0 semigroup $\{S(t)\}_{t\geq 0}$. The map (1.26) defines a linear bounded operator $B : U \to Z_0$. If $z_0 \in D(A)$ the generalized solution $z(\cdot, z_0)$ of (1.28a) starting in Z_0 is a continuous function $t \mapsto z(t, z_0) \in Z_0$ which can be represented in integral form as

$$z(t, z_0) = S(t)z_0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau \tag{1.29}$$

with

$$\| S(t) \| \leq \alpha\, e^{-\varkappa t}\,, \qquad \forall\, t > 0 \tag{1.30}$$

where α and $\varkappa$ are positive numbers, and which satisfies for any $u \in L^2(\mathbb{R}_+; U)$ the output relation

$$C \int_0^t S(t-\tau)Bu(\tau)\,d\tau = \int_0^t K(t-\tau)u(\tau)\,d\tau\,. \tag{1.31}$$

Note that if $z(t, z_0) \in D(A)$ for $t > 0$ then $z(\cdot, z_0)$ is an ordinary strong solution of (1.28a).

2 Solvability of the Riccati operator equation for the realizations of a class of Volterra equations

Let us assume that $F_1 = F_1^* \in \mathcal{L}(Y, Y)$, $F_2 \in \mathcal{L}(U, Y)$ and $F_3 = F_3^* \in \mathcal{L}(U, U)$ are bounded linear operators and introduce the bilinear form

$$\begin{aligned} j(x, y\,;\, u, v) := (F_1 x, y)_Y + (F_2 u, x)_Y + (F_2 v, y)_Y + (F_3 u, u)_U\,,& \\ \forall\, x, y \in Y\,,\ \forall\, u, v \in U\,.& \end{aligned} \tag{2.1}$$

A direct calculation shows that

$$j(x, y\,;\, u, v) = j(y, x\,;\, v, u),\ \forall\, x, y \in Y, \forall\, u, v \in U\,. \tag{2.2}$$

Introduce the linear operator

$$\mathcal{K}(u, h)\,(t) := (\mathcal{T} u)\,(t) + h(t)\ ,\ u \in L^2(\mathbb{R}_+; U)\,,\ h \in W^{1,2}_{-\rho}(\mathbb{R}_+; Y)\,,$$

and consider for $T > 0$ and a parameter ν, $|\nu| \leq \nu_0$, the bilinear functional

$$J^T_\nu(u, h) := \int_0^T \left[j(\mathcal{K}(u, h), \mathcal{K}(u, h); u, u) - \nu \| z(t, h, u) \|^2_{W^{1,2}_{-\rho}(\mathbb{R}_+;Y)} \right] dt\,, \tag{2.3}$$

which is for $\rho \in (0, \rho_0)$ a continuous map $L^2(0, T; U) \times W^{1,2}_{-\rho}(0, +\infty; Y) \to \mathbb{R}$.
The next theorem contains a version of the operator Riccati equation. For the case $U = Y = \mathbb{R}^n$ this theorem was proved in [7].

Theorem 2.1 *Let $\chi(p)$ be the Laplace transform of the absolutely continuous function $\mathbf{P}_\infty K$. Suppose that $F_1 = F_1^* \geq 0, F_3 = F_3^* > 0, F_3^{-1}$ exists,*

$$\chi(p) \in \mathcal{L}(U,Y), \forall\, p \in \mathbb{C},$$

and

$$\Pi(i\omega) := \chi^*(i\omega)F_1\chi(i\omega) + 2\,\mathrm{Re}(F_2^*\chi(i\omega)) + F_3 > 0, \quad \forall\, \omega \in \mathbb{R}. \tag{2.4}$$

Then there exists a sufficiently small $\nu_0 > 0$ such that for any $\nu \in [0,\nu_0]$ we have (the index ν is omitted):

1) For any $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ there exists a $\tilde{u}(h) \in L^2(0,+\infty;U)$ such that

$$J_0^T(\tilde{u}(h),h) < J_0^T(u,h)\,, \quad \forall\, u \in L^2(0,T;U)\,,\ \| u - \tilde{u}(h) \|_{L^2(0,T;U)} > 0\,.$$

2) There exists a bounded self-adjoint operator

$$\begin{aligned} M_T = M_T^* : W^{1,2}_{-\rho}(0,+\infty;Y) &\to W^{1,2}_{-\rho}(0,+\infty;Y) \\ (M_T h,h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} &= J_0^T(\tilde{u}(h),h), \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y)\,. \end{aligned}$$

3) In the case $T = \infty$ the operator $M := M_\infty$ satisfies the following Riccati operator equation

$$\begin{aligned} S(h,g) := (Ah,Mg)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (Mh,Ag)_{W^{1,2}_{-\rho}(0,+\infty;Y)} - (L^*h,L^*g)_U \\ + (F_1Ch,Cg)_Y - \nu(h,g)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \quad \forall\, h,g \in D(A)\,, \end{aligned} \tag{2.5}$$

where $N := \sqrt{F_3}, L := (MB + C^*F_2)N^{-1} \in \mathcal{L}(U, W^{1,2}_{-\rho}(0,+\infty;Y))$ and $(L^*h,v)_U = (h,Lv)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y), \forall\, v \in U.$

Proof Let us write the functional $J_0^T(u,h)$ with the help of the regular representation (1.23) – (1.28 b) as

$$J_0^T(u,h) = \int_0^T \left[(\mathbb{F}_1{}^\nu z,z)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + \ 2\,(\mathbb{F}_2 u,z)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (F_3\,u,u)_U\right] dt\,,$$

where $z = z(t,h,u)$ satisfies (1.28 a),

$$\mathbb{F}_1{}^\nu := C^*F_1C - \nu I_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,,$$

$$\mathbb{F}_2 := C^*F_2 \quad \text{and} \quad C^* \in \mathcal{L}(Y, W^{1,2}_{-\rho}(0,+\infty;Y)) \quad \text{is defined by}$$

$$(Ch,\xi)_U = (h,C^*\xi)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y), \quad \forall\, \xi \in U\,.$$

Because of (2.4) the coercivity condition for the principal quadratic part of the functional (2.3) is satisfied. This follows from the fact that the functional (2.3) can be written as

$$J_0^T(u,h) = \pi^T(u,u) - 2\,L_h^T(u) + J_0^T(0,h) \tag{2.6}$$

where

$$\pi^T(u,v) := \int_0^T \left[j(\mathcal{K}(u),\mathcal{K}(v);u,v) - \nu(z(u),z(v))_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt\,,$$

$$L_h^T(u) := -\int_0^T \left[(\mathbb{F}_1^\nu z(u), \bar z(h))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (\bar z(h), \mathbb{F}_2 u)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt\,. \tag{2.7}$$

In (2.7) we use the notation $z(u) := z(t,0,u)$ and $\bar z(h) := z(t,h,0)$.
If we denote by $\hat u_T(i\omega)$ the Fourier transform of the function $\mathbf{P}_T u(\cdot)$ we get by the Parseval equation

$$\pi^T(u,u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\Pi(i\omega)\hat u_T(i\omega), \hat u_T(i\omega)\right)_U d\omega$$
$$-\int_T^\infty \left(F_1\mathcal{K}(\mathbf{P}_T u),\, \mathcal{K}(\mathbf{P}_T u)\right)_Y - \nu\int_0^T \| z(u)\|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\, dt\,.$$

From (1.23), (2.4) and the condition imposed on F_1 it follows therefore that for sufficiently small $\nu_0 > 0$ we have for any real ν, $|\nu| \le \nu_0$, with a number $\varepsilon_1 > 0$ the inequality

$$\pi^T(u,u) \ge \varepsilon_1 \int_0^T |\,u(t)|_U^2\, dt\,, \quad \forall\, u \in L^2(0,T;U)\,. \tag{2.8}$$

Hence ([18]) we get the existence and uniqueness of an element $\tilde u \in L^2(0,T;U)$ such that

$$\inf_{u\in L^2(0,T;U)} J_0^T(u,h) = J_0^T(\tilde u,h)\,. \tag{2.9}$$

This $\tilde u$ is the unique solution in $L^2(0,T;U)$ of the Euler equation

$$\pi^T(\tilde u,u) = L_h^T(u), \quad \forall\, u\in L^2(0,T;U)\,.$$

The Euler equation has by virtue of (2.2), (2.6), (2.7) the form

$$\int_0^T \left[(\mathbb{F}_1^\nu \tilde z_h(t), z(u))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (\tilde z_h(t), \mathbb{F}_2 u)_{W^{1,2}_{-\rho}(0,+\infty;Y)} \right.$$
$$\left. + (z(u), \mathbb{F}_2\tilde u_h(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (F_3\tilde u_h(t), u(t))_U\right] dt = 0\,, \quad \forall\, u\in L^2(0,T;U)\,, \tag{2.10}$$

where $\tilde z_h(t) = z(t,h,\tilde u_h)$, $\tilde u_h(t) = \tilde u(h)(t)$. Thus the first assertion of the theorem is proven.
Consider now the function $\psi_h(\cdot) : [0,T) \to W^{1,2}_{-\rho}(0,+\infty;Y)$ given for $t\in[0,T)$ by

$$\psi_h(t) := \int_t^T S^*(s-t)[\,\mathbb{F}_1^\nu \tilde z_h(s) + \mathbb{F}_2\tilde u_h(s)\,]\, ds\,, \tag{2.11}$$

where $S^*(t)$ is the semigroup of linear bounded operators that are adjoint to $S(t)$. It is easy to see that

$$S^*(t)\phi = \tau^t\phi\,. \tag{2.12}$$

From (1.29), (2.11) and (2.12) it follows that for any test function ξ on $[0,T)$ such that $\xi(t) \in D(A)$ on $[0,T)$ and $\dot{\xi}(t) \in W^{1,2}_{-\rho}(0,+\infty;Y)$ we have

$$\int_0^T \left[(-\psi_h(t),\dot{\xi}(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (\psi_h(t), A\,\xi(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} \right.$$
$$\left. + (\mathbb{F}_1^\nu \tilde{z}_h(t) + \mathbb{F}_2\,\tilde{u}_h(t), \xi(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt + (\psi_h(0),\xi(0))_Y = 0\,. \tag{2.13}$$

Let us take in (2.13) as test function $\xi := z(\cdot,0,u)$ with $u \in L^2(0,T;U)$. This is possible according to the solutions properties of (1.28a). By transforming the obtained formula with the use of (1.28a) and (2.10) we obtain the equality

$$\int_0^T \left[(Bu(t),\psi_h(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (\mathbb{F}_2^* \tilde{z}_h(t) +\ F_3\tilde{u}_h(t), u(t))_U\right] dt = 0\,,$$
$$\forall\, u \in L^2(0,T;U)\,.$$

From this it follows that

$$B^*\psi_h(t) +\ F_3\tilde{u}_h(t) + \mathbb{F}_2^*\tilde{z}_h(t) = 0\,, \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y)\,,\ \forall\, t \in [0,T)\,. \tag{2.14}$$

Equation (2.14) shows that

$$\tilde{u}_h(t) = -F_3^{-1}[B^*\psi_h(t) + \mathbb{F}_2^*\tilde{z}_h(t)]\,.$$

In particular it follows from this that $\tilde{u}_h(\cdot)$ is continuous.
Now define the operator $M_T : W^{1,2}_{-\rho}(0,+\infty;Y) \to W^{1,2}_{-\rho}(0,+\infty;Y)$ by

$$M_T h := \psi_h(0)\,. \tag{2.15}$$

The map M_T is linear since it is the superposition of the linear maps

$$h \mapsto (\tilde{u}_h,\tilde{z}_h) \in L^2(0,T;U) \times W^{1,2}_{-\rho}(0,+\infty;Y) \mapsto \psi_h(0) \in W^{1,2}_{-\rho}(0,+\infty;Y)\,.$$

Let us show as in [7], that for $M := M_\infty$ the relation

$$M\tilde{z}_h(t) = \psi_h(t)\,, \quad \forall\, t > 0 \tag{2.16}$$

is true. For this we consider for $s \geq 0$ the parameter-dependent functional

$$J_s^\infty(u,h) := \int_s^\infty \left[(\mathbb{F}_1^\nu z_s, z_s)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right.$$
$$\left. + 2\,(\mathbb{F}_2 u, z_s)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (F_3 u,u)_U\right] dt\,, \tag{2.17}$$

where $\mathbb{F}_1^\nu, \mathbb{F}_2$ are as above and $z_s = z_s(t,h,u)$ is given by

$$z_s(t,h,u) = S(t-s)h + \int_s^t S(t-\tau)\,u\,(\tau)\,d\tau\,.$$

In the same way as above we can show the existence of an optimal control $\tilde{u}_{s,h}$ solving the minimization problem for the functional (2.17). Introduce now the associated functions

$$\tilde{z}_{s,h}(t) = z_s(t,h,\tilde{u}_{s,h})\,,$$
$$\psi_{s,h}(t) = \int_t^\infty S^*(\tau - t)\left[\mathbb{F}_1^\nu \tilde{z}_{s,h}(\tau) + \mathbb{F}_2\tilde{u}_{s,h}(\tau)\right] d\tau$$

As above we find a self-adjoint operator M^s satisfying

$$M^s \tilde{z}_{s,g}(s) = \psi_{s,g}(s)\,, \quad \forall\, g \in W^{1,2}_{-\rho}(0,+\infty;Y), s \geq 0\,. \tag{2.18}$$

Take in (2.18) the function $g := \tilde{z}_h(s)$. Then it follows from the fact, that the functional does not depend explicitely on the time, that

$$\tilde{z}_{s,g}(t) = \tilde{z}_{0,g}(t-s) = \tilde{z}_g(t-s) \quad \text{and}$$

$$\psi_{s,g}(t) = \psi_{0,g}(t-s)\,.$$

This implies that the operator M^s from (2.18) is independent on s. From Bellman's principle of dynamic programming ([3]) and the optimality of $\tilde{u}_h$ and $\tilde{u}_{s,h}$ it follows that

$$\begin{gathered} \tilde{z}_{s,g}(t) = \tilde{z}_h(t), \psi_{s,g}(t) = \psi_h(t)\,,\; t \geq s\,, \\ \text{where} \quad g = \tilde{z}_h(s)\,,\; h \in W^{1,2}_{-\rho}(0,+\infty;Y)\,. \end{gathered} \tag{2.19}$$

Using now (2.19) and the independence of M^s on s we get from (2.18) the property (2.16). If we take in (2.13) the function $\xi(t) := \tilde{z}_g(t)$ with $g \in D(A)$ and use the properties (2.2), (2.3), (2.15) and (2.16) we obtain the equality

$$\begin{aligned} (M_T h, g)_{W^{1,2}_{-\rho}(0,+\infty;Y)} &= \int_0^T \left[j\left(\mathcal{K}(\tilde{u}_h,h), \mathcal{K}(\tilde{u}_g,g)\,; \tilde{u}_h, \tilde{u}_g\right) - \nu(\tilde{z}_h(t), \tilde{z}_g(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt \\ &= (h, M_T g)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \qquad \forall\, h,g \in D(A)\,. \end{aligned} \tag{2.20}$$

Note that equation (2.20) can be extended to the entire space $W^{1,2}_{-\rho}(0,+\infty;Y)$. Let us prove this as in [18] it can be shown that if $h_m \to h$ strongly in $W^{1,2}_{-\rho}(0,+\infty;Y)$ then $\{\tilde{u}_{h_m}\}$ converges weakly in $L^2(0,T;U)$ to $\tilde{u}_h$. From this it follows that if there are two sequences $\{h_m\}, h_m \in D(A)$, and $\{g_k\}, g_k \in D(A)$, and two functions $h, g \in W^{1,2}_{-\rho}(0,+\infty;Y)$ which are the strong limits of the sequences $\{h_m\}$ and $\{g_k\}$, respectively, then by the continuity of the functional j with respect to u it follows that

$$\begin{aligned} &\lim_{m,k\to\infty} j\big(\mathcal{K}(\tilde{u}_{h_m}, h_m),\, \mathcal{K}(\tilde{u}_{g_k}, g_k)\,; \tilde{u}_{h_m}, \tilde{u}_{g_k}\big) \\ &\qquad = j\big(\mathcal{K}(\tilde{u}_h, h),\, \mathcal{K}(\tilde{u}_g, g)\,; \tilde{u}_h, \tilde{u}_g\big)\,. \end{aligned}$$

Thus the identity (2.20) is true in $W^{1,2}_{-\rho}(0,+\infty;Y)$. But this implies that the linear operator M_T defined on the entire $W^{1,2}_{-\rho}(0,+\infty;Y)$ is symmetric and, consequently, closed. By the closed graph theorem it follows that the operator M_T is continuous.
Let $T = \infty$ and consider an arbitrary function $h \in D(A)$. It follows that $\tilde{z}_h(t) \in D(A)$ for all $t > 0$ and, using the continuity of $\tilde{u}_h(\cdot)$, that $A\tilde{z}_h(\cdot) : (0,\infty) \to W^{1,2}_{-\rho}(0,+\infty;Y)$ is continuous. Thus we have the relation

$$\begin{aligned} \int_0^\infty & \left[(A\tilde{z}_h + B\tilde{u}_h, M\xi)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (\tilde{z}_h, M\dot{\xi})_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt \\ &\qquad - (h, M\xi(0))_{W^{1,2}_{-\rho}(0,+\infty;Y)} = 0\,, \end{aligned}$$

which we can add to (2.13). If we use (2.14), (2.16) and the symmetry of M we obtain the relation

$$\int_0^\infty S(\tilde{z}_h(t), \xi(t))\, dt = 0\,, \quad \forall\, h, \xi \in D(A)\,, \dot{\xi} \in W^{1,2}_{-\rho}(0, +\infty; Y)\,.$$

Since ξ and h can be taken arbitrary in the associated spaces, we obtain the identity (2.5). ■

Corollary 2.1 *Suppose that for (1.20) and the associated state-space realization (1.28a), (1.28b) the conditions of Theorem 2.1 are satisfied. Then for sufficiently small $\nu > 0$, any $h \in D(A)$ and any continuous function $u \in L^2(0, \infty; U)$ the pair $(z(\cdot), y(\cdot))$, where $z(\cdot) \equiv z(\cdot, h, u)$ is the solution of (1.28a) with $z(0, h, u) = h$ and $y(\cdot) = Cz(\cdot)$, satisfy for $t > 0$ the relation*

$$\begin{aligned}&\frac{d}{dt}(Mz(t), z(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} = \| L^* z(t) + N u(t) \|^2_U \\ &- \left[(F_1 y(t), y(t))_Y \,+\, 2\,(F_2 u(t), y(t))_Y \,+ (F_3 u(t), u(t))_U\right] - \nu \,\| z(t) \|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,. \quad (2.21)\end{aligned}$$

Here $M = M^*, L$ and N are the operators from part 3) of Theorem 2.1.

Proof Since $h \in D(A)$ we can assume that $z(\cdot)$ is a strong solution of (1.28a). Thus we can write, using the equation (2.5),

$$\begin{aligned}&\frac{d}{dt}(Mz(t), z(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} = \;2\;(Mz(t), \dot{z}(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} \\ &= \;2\;(Mz(t), Az(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + 2\;(Mz(t), Bu(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} \\ &= \;2\;(Mz(t), Bu(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + \| L^* z(t) \|^2_U - \;(F_1 C z(t), Cz(t))_Y + \nu \,\| z(t) \|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,. \end{aligned}$$
(2.22)

Now we use again (2.5) to get the expression

$$\begin{aligned}&(L^* z(t) + Nu(t), L^* z(t) + Nu(t))_U = (L^* z(t), L^* z(t))_U \\ &+\; 2\,(Mz(t), Bu(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + 2\;(C^* F_2 u(t), z(t))_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (F_3 u(t), u(t))_U\,.\end{aligned}$$
(2.23)

Putting now $\|L^* z(t)\,\|^2_U$ from (2.23) into (2.22) the formula (2.21) follows immediately. ■

3 Stability and instability of infinite-dimensional Volterra equations by their state-space realizations

Consider the Volterra integral equation

$$y(t) = h(t) + \int_0^t K(t - \tau)\, \varphi\,(y(\tau), \tau)\, d\tau\,, \qquad (3.1)$$

where $K(t) \in \mathcal{L}(U,Y)(U,Y$ Hilbert spaces) is twice piecewise-differentiable satisfies (1.21) and (1.22), and has therefore a state-space realization (1.23) - (1.28b). Suppose that

$$\varphi : Y \times \mathbb{R}_+ \to U \tag{3.2}$$

is a continuous function.
Instead of one fixed nonlinearity φ we consider a family $\mathcal{N}$ of continuous maps (3.2), such that for any $\varphi \in \mathcal{N}$ and any $h \in D(A)$ with $D(A)$ from (1.24) the nonlinear integral equation (3.1) has a unique solution $y(\cdot,h,\varphi)$ and this solution is continuous. Suppose also that there are linear bounded operators $G_1 = G_1^* \in \mathcal{L}(Y,Y), G_1 \leq 0, G_2 \in \mathcal{L}(U,Y)$ and $G_3 = G_3^* \in \mathcal{L}(U,U), G_3 < 0, G_3^{-1}$ exists, such that for any $\varphi \in \mathcal{N}$ we have

$$(G_1 y, y)_Y + 2\,(G_2\varphi(y,t), y)_Y + (G_3\varphi(y,t), \varphi(y,t))_U \geq 0, \quad \forall\, t \geq 0,\ \forall\, y \in Y\,. \tag{3.3}$$

Now we consider together with the state-space equation (1.28 a), (1.28 b) and the nonlinearity $\varphi \in \mathcal{N}$ the nonlinear evolution system

$$\dot z = Az + B\varphi(y,t),\ y = Cz\,. \tag{3.4}$$

From the uniqueness of continuous solutions for (3.1), (3.2) with $h \in D(A)$ it follows immediately, since (3.1) is the integral representation of (3.4), that (3.4) has for any $\varphi \in \mathcal{N}$ and any initial function $h \in D(A)$ a unique solution $z(\cdot,h,\varphi)$ on $\mathbb{R}_+$ with $z(0,h,\varphi) = h$. In the next theorem we show the connection between solutions of the integral equation (3.1) and the solutions of the associated state-space realizations (3.4).

Theorem 3.1 *Suppose that the following conditions are satisfied:*

- *a) Let $\chi(\cdot)$ be the Laplace transform of $\mathbf{P}_\infty K$ and let with the operators $F_1 = -G_1$, $F_2 = -G_2$ and $F_3 = -G_3$ from (3.3) the frequency-domain condition (2.4) be true;*
- *b) For any $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ and $\varphi \in \mathcal{N}$ the solution $y(\cdot) = y(\cdot,h,\varphi)$ of (3.1) exists and is continuous.*

Then for any $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ and $\varphi \in \mathcal{N}$ the solution $z(\cdot) = z(\cdot,h,\varphi)$ of (3.4) with the initial condition $z(0) = h$ exists and there are a bounded linear self-adjoint operator $P : W^{1,2}_{-\rho}(0,+\infty;Y) \to W^{1,2}_{-\rho}(0,+\infty;Y)$ and a constant $\delta > 0$ such that for any $t_1, t_2 \geq 0, t_1 < t_2$, we have:

$$(Pz(t,h,\varphi), z(t,h,\varphi))_{W^{1,2}_{-\rho}(0,+\infty;Y)}\Big|_{t_1}^{t_2} \leq$$
$$-\,\delta \int_{t_1}^{t_2} \left[\| \varphi(y(t,h,\varphi),t) \|_U^2 + \| z(t,h,\varphi) \|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt\,, \quad \forall\, \varphi \in \mathcal{N}, \quad \forall\, h \in D(A)\,. \tag{3.5}$$

Proof Let us assume for a moment that the kernel $K(\cdot)$ satisfies the conditions from Section 2 and $h \in D(A)$. Introduce the continuous function $u(t) := \varphi(y(t,h,\varphi),t)$ for

$t > 0$. With this function we can applicate Corollary 2.1. The integration of (2.21) on an arbitrary time interval $0 \le t_1 < t_2$ with $P := -M$ and $F_i := -G_i, i = 1, 2, 3$, gives

$$\begin{aligned} &\big(Pz(t,h,\varphi), z(t,h,\varphi)\big)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\Big|_{t_1}^{t_2} \\ &\le \int_{t_1}^{t_2} \big[\big(G_1 y(t,h,\varphi), y(t,h,\varphi)\big)_Y + 2\,(G_2 u(t), y(t,h,\varphi))_Y \\ &+ (G_3 u(t), u(t))_U\big]\,dt - \nu \int_{t_1}^{t_2} \| z(t,h,\varphi) \|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,dt \end{aligned} \tag{3.6}$$

It follows from (1.31) and the boundedness of the operator P that the left- and right-hand side of (3.6) depend continuously on $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ and $K \in \mathcal{L}(U,Y)$.
From this and the denseness of $D(A)$ in $W^{1,2}_{-\rho}(0,+\infty;Y)$ it follows that the inequality (3.6) can be continued for functions $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$.
Since the inequality (2.4) is strictly we can get a similar inequality (3.6) with $\widetilde{G}_3 = G_3 - \delta_1 I$ where $\delta_1 > 0$ is sufficiently small. This modified inequality (3.6) and (3.3) immediately give (3.5). ■

In order to describe the absolute stability or instability behaviour of (3.1) with the help of the state-space realization (3.4) we need an additional assumption on the class $\mathcal{N}$. Let us assume that there is a linear bounded operator $R : Y \to U$ such that the "nonlinearity" $\varphi = Ry$ belongs to $\mathcal{N}$.

Theorem 3.2 *Suppose that $\chi(\cdot)$ is the Laplace transform of $\mathbf{P}_\infty K$ and the operator function $(I - \chi(p)R)^{-1}$ has poles in the right half-plane and the frequency-domain condition (2.4) is satisfied with $F_i = -G_i, i = 1,2,3$. Then there exists a bounded linear self-adjoint operator*

$$P : W^{1,2}_{-\rho}(0,+\infty;Y) \to W^{1,2}_{-\rho}(0,+\infty;Y) \qquad \textit{such that}$$
$$\mathcal{C} := \{h \in W^{1,2}_{-\rho}(0,+\infty;Y) \mid (Ph,h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} < 0\}$$

is a quadratic cone $\mathcal{C} \notin \emptyset$ in $W^{1,2}_{-\rho}(0,+\infty;Y)$ with the following properties:
a) There exists a constant $\beta > 0$ such that for any $h \in \mathcal{C}$ and any $\varphi \in \mathcal{N}$

$$\lim_{t\to\infty} e^{-\beta t} \int_0^t \| \varphi(y(s,h,\varphi),s) \|^2_U\,ds = \infty\,. \tag{3.7}$$

b) Any solution $y(\cdot,h,\varphi)$ of (3.1) which does not satisfy (3.7) has the property $\int_0^\infty \| \varphi(y(s,h,\varphi),s)\|^2_U\,ds < \infty$ and, consequently,

$$\varphi(y(\cdot,h,\varphi),\cdot) \in L^2(0,\infty;U) \quad \textit{and} \quad y(\cdot,h,\varphi) \in L^2(0,\infty;Y)\,. \tag{3.8}$$

Proof Let us show that $\mathcal{C} \notin \emptyset$. Assume the contrary, i.e. assume that $(Ph,h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} \ge 0,\ \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y)$. From (3.6) we obtain for any $t > 0$, any $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ and $\varphi \in \mathcal{N}$ that

$$\begin{aligned} (Pz(t,h,\varphi), z(t,h,\varphi))_{W^{1,2}_{-\rho}(0,+\infty;Y)} &\le (Pz_0, z_0)_{W^{1,2}_{-\rho}(0,+\infty;Y)} \\ &- \delta \int_0^t \| \varphi(y(s,h,\varphi),s) \|^2_U\,dt\,. \end{aligned} \tag{3.9}$$

It follows that $\varphi(y(\cdot,h,\varphi),\cdot) \in L^2(0,\infty;U), \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y), \forall\, \varphi \in \mathcal{N}$. But the last property is impossible since by assumption for the nonlinearity $\tilde{\varphi} := Ry \in \mathcal{N}$ there exists a $\tilde{h} \in W^{1,2}_{-\rho}(0,+\infty;Y)$ such that $\tilde{\varphi}(\tilde{y}(\cdot,h,\tilde{\varphi}) = R\tilde{y}(\cdot,\tilde{h},Ry)$ does not belong to $L^2(0,\infty;U)$.

Thus $\mathcal{C} \notin \emptyset$. Suppose $h \in \mathcal{C}$. Consider for arbitrary $\varphi \in \mathcal{N}$ the function $V(t) := (-Pz(t,h,\varphi), z(t,h,\varphi))_{W^{1,2}_{-\rho}(0,+\infty;Y)}$. It follows from (3.9) and the boundedness of the operator $-P$ that there exists a $\lambda > 0$ such that

$$\dot{V}(t) - \lambda V(t) \geq 0\,, \quad \forall\, t \geq 0\,. \tag{3.10}$$

The integration of (3.10) on $[0,t]$ gives the inequality

$$V(t) \geq e^{\lambda t}\, V(0)\,, \quad \forall t \geq 0\,. \tag{3.11}$$

It follows that for any $\beta \in (0,\lambda)$ the property $V(t)e^{-\beta t} \to +\infty$ for $t \to +\infty$ holds. From the boundedness of P we conclude that there are constants $c_1 > 0, c_2 > 0$ such that

$$V(t) \leq c_1 \|\, z(t,h,\varphi)\,\|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}$$

and

$$\|\, z(t,h,\varphi)\,\|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)} \leq c_2 \int_0^t \|\, \varphi(y(s,h,\varphi),s)\,\|^2_U\, ds\,, \quad \forall t \geq 0\,.$$

Suppose that $(Ph,h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} \geq 0\,, \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y)$. Then it follows from (3.9) that on an arbitrary time interval $[0,t]$ we have

$$\delta \int_0^t \left[\|\varphi(y(s,h,\varphi),s)\|^2_U + \|z(t,h,\varphi)\|^2_{W^{1,2}_{-\rho}(0,+\infty;Y)}\right] dt \\ \leq V(0) - V(t) \leq V(0)\,.$$

But this implies that $\varphi(y(\cdot,h,\varphi),\cdot) \in L^2(0,\infty,U)$.

■

4 PDE's with boundary control and ODE's with delay as realizations of the same Volterra equation

Let us investigate the question how to suppress vibrations in a fluid conveying tube. We consider for this a system of equations which is described in [8, 15, 19, 23, 27].

The motion of an incompressible fluid is given for $t > 0$ in the acoustic approximation by

$$\frac{\partial v}{\partial t} = a_1 \frac{\partial w}{\partial x}\,, \qquad \frac{\partial w}{\partial t} = a_2 \frac{\partial v}{\partial x}\,,\ x \in (0,1)\,, \tag{4.1}$$

where a_1 and a_2 are positive parameters, v denotes the relative velocity of the fluid and w denotes the pressure. The boundary conditions are given for $t > 0$ by

$$w(t,1) = 0, \qquad \left(\frac{1}{2}w(t,0) - v\,(t,0)\right) = -u\,(t)\,, \tag{4.2}$$

where $u(\cdot)$ is a function ("boundary control") which describes the relative displacement of the piston of a servomotor. The equation of the turbine for $t > 0$ is

$$T_a \frac{dq}{dt} + q(t) = u(t) + \frac{3}{2} w(0,t) \,. \tag{4.3}$$

Here q denotes the relative angular speed of the turbine, T_a is a positive parameter. The regulator is described by the equation

$$T_r^2 \frac{d^2\zeta}{dt^2} + T_k \frac{d\zeta}{dt} + \delta\zeta + k\varphi(\dot{\zeta}) + q(t) = 0 \,, \tag{4.4}$$

where ζ represents the displacement of the clutch of the regulator and T_r, T_k, δ and k are positive parameters. The friction term is given by a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ defined through a parameter $\kappa > 0$ by

$$\varphi(y) = \begin{cases} 1 & \text{if} \quad y \geq \kappa \,, \\ \frac{1}{\kappa} y & \text{if} \quad y \in (-\kappa, \kappa) \,, \\ -1 & \text{if} \quad y \leq -\kappa \,, \end{cases}$$

and thus satisfying the property

$$\varphi(y)\, y \geq 0 \,, \quad \forall\, y \in \mathbb{R} \,. \tag{4.5}$$

The equation of the servomotor is

$$T_s \frac{du}{dt} = \eta(t) \,, \tag{4.6}$$

where T_s is a positive parameter and η denotes the displacement of the slide value. The last condition is for $t > 0$ and with a positive parameter β

$$\eta(t) - \zeta(t) + \beta u(t) = 0 \,. \tag{4.7}$$

A direct computation shows ([15, 27]) that the transfer function of the linear part of (4.1) – (4.7) which connects the (formal) Laplace transforms of $-\varphi$ and $\dot{\zeta}$ is given by

$$\chi(p) = k \frac{p\,(T_a p + 1)\,(T_s p + \beta)\,(\sinh p\tau + \alpha \cosh p\tau)}{(T_r^2 p^2 + T_k p + \delta)\, Q(p) + R(p)} \,, \tag{4.8}$$

where

$$\alpha = 2\sqrt{\frac{a_1}{a_2}} \,, \quad \tau = 1/\sqrt{a_1 a_2} \,, \tag{4.8 a}$$

$$Q(p) = (T_s p + \beta)\,(T_a p + 1)(\alpha \cosh p\tau + \sinh p\tau) \,,$$

$$R(p) = 2\cosh p\tau - 2\sinh p\tau \,.$$

Note that $\chi(p)$ can be written with some $c > 0$ as

$$\chi(p) = \frac{k}{T_r^2 p + c} + \chi_1(p) \,. \tag{4.9}$$

where

$$\chi_1(p) = \frac{k\,(c-T_k)\,p\,Q(p) - \delta Q(p) - R(p)}{(T_r^2\,p + c)P(p)}\,. \tag{4.10}$$

The representation (4.10) shows that $\chi_1(p)$ is analytic in some halfplane $\{\mathrm{Re}\,p > -\varepsilon\}$ with $\varepsilon > 0$. From this it follows that $\chi_1(p)$ has a Laplace original $K_1(t)$ which is absolute continuous, satisfies the inequalities

$$|\,K_1(t)\,| \le \mathrm{const}\,e^{-\varepsilon_0 t} \tag{4.11}$$

with some $\varepsilon_0 > 0$ and such that K_1 and $\dot{K}_1$ belong to $L^2(0,\infty;\mathbb{R})$. Since first part of (4.9) has the Laplace original $\frac{k}{T_r^2}e^{-ct/T_r^2}$ the whole original of $\chi(p)$ can be represented as $K(t) = K_1(t) + \frac{k}{T_r^2}e^{-ct/T_r}$. It is shown in [15, 27] that any solution component $y(t) := \dot{\zeta}(t)$ from (4.4) can be written as

$$y(t) = h(t) + \int_0^t K(t-\tau)\varphi(y(\tau))\,d\tau\,, \tag{4.12}$$

where again h is absolute continuous, satisfies an inequality of type (4.11) and $h, \dot{h}$ belong to $L^2(0,\infty;\mathbb{R})$.
The quadratic contraints (3.3) can be described in $Y = U = \mathbb{R}$ by the inequality

$$\varphi(y)\,(y - \kappa\,\varphi(y)) \ge 0\,, \quad \forall\, y \in \mathbb{R}\,, \tag{4.13}$$

i.e., (3.3) is satisfied with $G_1 = 0, G_2 = \frac{1}{2}$ and $G_3 = -\kappa < 0$.
Using the transfer function (4.9) and the constraints (4.13) we can verify the frequency-domain condition (2.4). A direct computation shows (see [27, 15]) that if

$$T_k(\alpha^2-1)\beta^2 \le \left(\frac{55}{32} + \alpha^2\right)(\beta\,T_a + T_s) \tag{4.14}$$

is satisfied and then the condition

$$\alpha(T_k\beta^2 - (\beta\,T_a + T_s)) \ge 3\,\tau\beta\,. \tag{4.15}$$

is necessary and sufficient for the frequency-domain condition (2.4). The stability and instability domains of the denominator of $\chi(p)$ where investigated in [19] and characterized in the (T_k, T_r^2)-plane by domains Ω_{st} and Ω_{unst}, respectively. It follows now from Theorem 3.1 that under the conditions (4.13) - (4.15) for parameters from Ω_{st} the solutions of the integral equation (4.12) have the properties described by Theorem 3.1.

In the special case $a_1 = a_2 =: a > 0$ equation (4.1) is the wave equation $\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2}$ and condition (4.15) is satisfied if $T_k\beta^2 - (\beta T_a + T_s) \ne 0$ and

$$a \ge \frac{3\,\beta}{2\,(T_k\beta^2 - (\beta\,T_a + T_s))}\,,$$

i.e. if the speed of sound a is sufficiently large.

The hybrid system (4.1) – (4.7) consisting of a hyperbolic PDE and ODE's is one of the possible realizations of the input-output process given by the Volterra integral equation (4.12) with the class of nonlinearities described by (4.5) and a class of kernels characterized by their Laplace transforms (4.9). Let us show that we can also choose a realization for (4.12), (4.5), (4.9) in the form of a hybrid system consisting of an ODE with delay and algebraic equations with delay. Note that such a realization has some advantages in practice. Using such a delay system with the same stability properties as the PDE realization we can reduce the amount for computation and stability analysis. In synchronization theory the introduction of delay interactions instead of wave interactions is demonstrated in [24].

Introduce in (4.1) with α from (4.8a) the new functions

$$\tilde{v} := v - \alpha/2\, w \qquad \text{and} \qquad \tilde{w} := v + \alpha/2\, w\,. \tag{4.16}$$

A straightforward computation shows that these functions satisfy the relations

$$\frac{\partial \tilde{v}}{\partial t} + \frac{1}{\tau}\frac{\partial \tilde{v}}{\partial x} = 0\,, \quad \frac{\partial \tilde{w}}{\partial t} - \frac{1}{\tau}\frac{\partial \tilde{w}}{\partial x} = 0\,, \quad x \in (0,1), t > 0\,, \tag{4.17}$$

where τ is again given by (4.8a).

The new boundary conditions can be written for $t > 0$ as

$$\tilde{v}(t,0) + \alpha_1 \tilde{w}(t,0) = \nu_1(t) \tag{4.18}$$

with

$$\alpha_1 := \frac{\alpha - 1}{\alpha + 1}\,, \qquad \nu_1(t) := \frac{2\,\alpha}{1+\alpha} u(t)\,, \tag{4.19}$$

and

$$\tilde{v}(t,1) + \tilde{w}(t,1) = 0\,. \tag{4.20}$$

The ODE's (4.3), (4.4), (4.6) together with the algebraic relation (4.7) transform into the system

$$\left.\begin{array}{rcl}
\dot{q}(t) &=& -\frac{1}{T_a} q(t) + \frac{1}{T_a} u(t) - \frac{3}{2\,T_a\alpha}\tilde{v}(0,t) + \frac{3}{2\,T_a\alpha}\tilde{w}(0,t)\,,\\
\dot{\zeta}(t) &=& \Theta(t)\,,\\
\dot{\Theta}(t) &=& -\frac{1}{T_r^2} q(t) - \frac{\delta}{T_r^2}\,\zeta(t) - \frac{T_k}{T_r^2}\Theta(t) - \frac{k}{T_r^2}\varphi(\Theta(t))\,,\\
\dot{u}(t) &=& \frac{1}{T_s}\,\zeta(t) - \frac{\beta}{T_s} u(t)\,.
\end{array}\right\} \tag{4.21}$$

Note that the hybrid system (4.17) - (4.21) is a special case of the system (2.1) considered in [23] with

$$\left.\begin{array}{rcl}
\tau(\lambda) &\equiv& \frac{1}{\tau}\,,\ \alpha_1 \text{ and } \nu_1(t) \text{ from (4.19)},\ \nu_2(t) \equiv 0\,,\ \beta_1 = \beta_2 = 0\,,\\
\nu(t) &=& \Theta(t)\,,\ \alpha_2 = -1\,, c_0^T = (0,0,1,0)\,,\\
c_1^T &=& \left(0,0,0,\frac{2\,\alpha}{1+\alpha}\right)\,,\ c_2^T = (0,0,0,0)\,,\\
b_{11}^T &=& \left(-\frac{3}{2\,T_a\alpha}\,,0,0,0\right)\,,\ b_{12}^T = \left(\frac{3}{2\,T_a\alpha}\,,0,0,0\right)\,,\\
b_{21}^T &=& (0,0,0,0)\,,\ b_{22}^T = (0,0,0,0)\,,\\
b_0^T &=& (0,0,\frac{k}{T_r^2}\,,0)\,,
\end{array}\right\} \tag{4.22}$$

$$A = \begin{pmatrix} -\frac{1}{T_a} & 0 & 0 & \frac{1}{T_a} \\ 0 & 0 & 1 & 0 \\ -\frac{1}{T_r^2} & -\frac{\delta}{T_r^2} & -\frac{T_k}{T_r^2} & -\frac{k}{T_r^2} \\ 0 & \frac{1}{T_s} & 0 & -\frac{\beta}{T_s} \end{pmatrix} . \tag{4.23}$$

Using the functions, vectors and matrices given by (4.22), (4.23) we define the hybrid system consisting of an ODE with delay and an algebraic relation with delay by

$$\left.\begin{aligned} \dot{z}(t) &= (A + b_{11}c_1^T)z(t) + (b_{12} - \alpha_1 b_{11})\,\eta_2\,(t-\tau) - b_0\varphi\,(y(t))\,, \\ \eta_1(t) &= c_1^T\,z(t) - \alpha_1\,\eta_2(t-\tau)\,, \\ \eta_2(t) &= \eta_1\,(t-\tau)\,, \\ y(t) &= c_0^T z\,(t)\,. \end{aligned}\right\} \tag{4.24}$$

Note that in the case $a_1 = a_2 =: a > 0$ in (4.1) the speed of sound a defines the delay time $\tau = \frac{1}{a}$ in (4.24). A direct computation shows that the transfer function of the linear part of (4.24) coincides with the function $\chi(p)$ given by (4.9), (4.10). From the results in [23], § 16, it follows that system (4.24) can be written as Volterra integral equation (4.12) with forcing function $h(t)$ and kernel $K(t)$ computed in [23], § 16. Thus we can consider equation (4.24) as a further realization of the input-output process represented by the Volterra integral equation (4.12). It follows by Theorem 3.1 that system (4.24) has the same stability properties as the first realization of (4.12), given by (4.1) – (4.7).

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