# Convergence in evolutionary variational inequalities with hysteresis nonlinearities

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## 1. Basic notation

$Y_0$ real Hilbert space, $(\cdot,\cdot)_0$ scalar product, $\|\cdot\|_0$ norm,

$\Lambda \in \mathcal{L}(\mathcal{D}(\Lambda), Y_0)$ self-adjoint, $\mathcal{D}(\Lambda) \subset Y_0$ dense,

$$(\Lambda y, \Lambda y)_0 \geq \|y\|_0^2 \,, \quad \forall\, y \in \mathcal{D}(\Lambda)$$

$Y_1 := \mathcal{D}(\Lambda)$ Hilbert space with scalar product

$$(y,\eta)_1 := (\Lambda y, \Lambda \eta)_0 \,, \forall\, y, \eta$$

New scalar product in $Y_0$

$$(y,\eta)_{-1} := (\Lambda^{-1} y, \Lambda^{-1}\eta)_0 \,, \; \forall\, y, \eta \in Y_0 \,,$$

$Y_{-1}$ completion of $Y_0$ w.r.t. $(\cdot,\cdot)_{-1}$

$\Rightarrow \; Y_1 \subset Y_0 \subset Y_{-1}$ dense and continuous embedding

("rigged Hilbert space structure")

$|(\eta, y)_0| = |(\Lambda^{-1}\eta, \Lambda y)_0| \leq \|\Lambda^{-1}\eta\|_0 \|\Lambda y\|_0 = \; \|\eta\|_{-1}\|y\|_1 \,,$

$\forall\, y \in Y_1, \; \eta \in Y_0$

$(\cdot,\cdot)_{-1,1}$ extension by continuity of the functionals $(\cdot, y)_0$ onto $Y_{-1}$

$(\cdot,\cdot)_{-1,1}$ coincides with $(\cdot,\cdot)_0$ on $Y_0 \times Y_1$ and satisfies

$|(\eta, y)_{-1,1}| \leq \|\eta\|_{-1}\|y\|_1 \,, \; \forall\, \eta \in Y_{-1}, y \in Y_1$

$-\infty \leq T_1 < T_2 \leq +\infty \,, \; L^2(T_1, T_2; Y_j) \,, \; j = 1, 0, -1$

Bochner measurable functions with

$$\|y\|_{2,j} := \left( \int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt \right)^{1/2}$$

$\mathcal{W}(T_1, T_2)$ space of functions s.th.

$y \in L^2(T_1, T_2; Y_1), \dot{y} \in L^2(T_1, T_2; Y_{-1})$ equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2)} := (\|y\|^2_{2,1} + \|\dot{y}\|^2_{2,-1})^{1/2}$$

Lions embedding theorem: Each function from $\mathcal{W}(T_1, T_2)$ belongs to $C(T_1, T_2; Y_0)$.

## 2. Evolutionary variational inequalities

$\Xi, Z$ real Hilbert spaces with scalar products $(\cdot,\cdot)_\Xi, (\cdot,\cdot)_Z$ and norms $\|\cdot\|_\Xi$ , $\|\cdot\|_Z$

$A \in \mathcal{L}(Y_1, Y_{-1})$ , $B \in \mathcal{L}(\Xi, Y_{-1})$ , $C \in \mathcal{L}(Y_0, \Xi)$

$\varphi : \mathcal{D}(\varphi) \subset W^{1,2}(0, T; Z) \times \Xi \to W^{1,2}(0, T; \Xi)$

strongly continuous hysteresis operator

$\mathcal{E} : Z \to 2^\Xi$ , $\mathcal{D}(\varphi) = \{(z, \xi_0) \in W^{1,2}(0, T; Z) \times \Xi \mid$

$\xi_0 \in \mathcal{E}(z(0))\}$

$\psi : Y_1 \to \mathbb{R} \cup \{+\infty\}$ convex, lower semi-continuous, $\psi \not\equiv +\infty$

$$\begin{aligned}&(\dot{y}(t) - Ay(t) - B\xi(t),\ \eta - y(t))_{-1,1} \\ &+ \psi(\eta) - \psi(y(t)) \geq 0\,, \quad \forall\, \eta \in Y_1 \quad \text{a.e.} \quad t \in (0, T)\end{aligned} \tag{1}$$

$$\xi(t) = \varphi(z, \xi_0)(t),\ z(t) = Cy(t), \tag{2}$$

$y(0) = y_0 \in Y_0,\ \xi_0 \in \mathcal{E}(z(0))$

A pair of functions $\{y, \xi\} \in \mathcal{W}(0, T) \cap C(0, T); Y_0) \times L^2(0, T; \Xi)$

which satisfies (1), (2), is called *solution* of (1), (2) on $(0, T)$

with initial conditions $y(0) = y_0$ and $\xi(0) = \xi_0$ ;

$y(\cdot)$ is the *state trajectory*, $\xi(\cdot)$ is the *control*.

A solution $\{y, \xi\}$ of (1), (2) (and the associated state trajectory $y$ and the control $\xi$) with $y(t) \equiv \widehat{y} \in Y_1$ and $\xi(t) \equiv \widehat{\xi} \in \Xi$ are called *stationary*. The set of all stationary solutions of (1), (2) is the *stationary set* $S$.

Any stationary solution $\{\widehat{y}, \widehat{\xi}\,\}$ satisfies the stationary variational inequality

$$(-A\widehat{y} - B\widehat{\xi}, \eta - \widehat{y}\,)_{-1,1} + \psi(\eta) - \psi(\widehat{y}\,) \geq 0\,, \quad \forall\, \eta \in Y_1 \quad (3)$$

$$\widehat{\xi} = \varphi(\widehat{z}, \widehat{\xi}\,)\;, \quad \widehat{z} = C\widehat{y}\,, \quad \widehat{\xi} \in \mathcal{E}(\widehat{z}\,) \quad (4)$$

**Special case**: $\psi(y) \equiv 0 \quad \Rightarrow$ (1), (2) is a *variational equation*

$$\dot{y} = Ay + B\xi(t)\,, \qquad (1')$$

$$\xi(t) = \varphi(z(t),\, \xi_0)(t), \quad \xi_0 \in \mathcal{E}(z(0)) \qquad (2')$$

Any stationary solution $\{\widehat{y}, \widehat{\xi}\,\}$ of $(1'), (2')$ is given by

$$0 = A\widehat{y} + B\widehat{\xi}\;, \quad \widehat{\xi} = \varphi(\widehat{z}, \widehat{\xi}\,)\,. \qquad (3'), (4')$$

If $A^{-1}$ exists, $(3'), (4')$ is equivalent to

$$S = \Big\{ (\widehat{y}, \widehat{\xi}\,) \in Y_1 \times \Xi \;\Big|\; \widehat{y} = -A^{-1}B\widehat{\xi}\;, \quad C\widehat{y} + CA^{-1}B\widehat{\xi} = 0\,,$$

$$\widehat{\xi} = \varphi(\widehat{z}, \widehat{\xi}\,) \Big\} =$$

$$\Big\{ (\widehat{y}, \widehat{\xi}\,) \in Y_1 \times \Xi \;\Big|\; \widehat{y} = -A^{-1}B\widehat{\xi},\; \widehat{z} + \chi(0)\widehat{\xi} = 0,\; \widehat{\xi} = \varphi(\widehat{z}, \widehat{\xi}) \Big\}\,,$$

$\chi(s) = C(A - sI)^{-1}B$ transfer operator function.

## 3. Convergence to the stationary set

**(A0)** The inequality (1), (2) has for arbitrary $y_0 \in Y_0$ and $\xi_0 \in \mathcal{E}(Cy_0(0))$ at least one solution $\{y, \xi\}$.

The stationary set $S$, given by (3),(4), is non-empty.

**(A1)** $\exists\, F_1 \in \mathcal{L}(Z, \Xi) \qquad \exists\, F_2 \in \mathcal{L}(\Xi, \Xi)$

$\forall\, T \geq 0 \quad \forall\, z \in W^{1,2}(0,T;Z) \quad \forall\, \xi_0 \in \mathcal{E}(z(0))$ :

$$\int_0^T \big(\dot\varphi(z,\xi_0)(t), F_1\dot z(t)\big)_\Xi - \big(\dot\varphi(z,\xi_0)(t), F_2\dot\varphi(z,\xi_0)(t)\big)_\Xi dt \geq 0.$$

**(A2)** $\exists\, \varkappa \in \{-1, 1\} \quad \exists\, G_1 \in \mathcal{L}(\Xi, Z) \quad \forall\, T \geq 0$

$\forall\, z \in W^{1,2}(0,T;Z) \quad \forall\, \xi_0 \in \mathcal{E}(z(0)) \quad \exists\, \gamma(z(0)) \geq 0$

$$\varkappa \int_0^T \big(G_1\varphi(z,\xi_0)(\tau), \dot z(\tau)\big)_Z \, d\tau \geq -\gamma(z(0)).$$

**(A3)** The pair $(A, B)$ is $L^2$-controllable, i.e. $\forall\, y_0 \in Y_0$

$\exists\, \xi(\cdot) \in L^2(0,+\infty;\Xi)$ such that

$$\dot y = Ay + B\xi, \quad y(0) = y_0\,,$$

is well-posed in the variational sense on $(0, +\infty)$.

**(A4)** Any solution of $\dot y = Ay,\ y(0) \in Y_0$, is exponentially decreasing for $t \to +\infty$.

**(A5)** The operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is regular, i.e.

$\forall\, T > 0, y_0 \in Y_1, w_T \in Y_1$ and $\forall\, f \in L^2(0,T;Y_{-1})$

the solutions of

$$\dot{y} = Ay + f(t), \quad y\,(0) = y_0\;, \quad \text{and of}$$

$$\dot{w} = -A_w^+ + f(t), \quad w(T) = w_T\;,$$

are strongly continuous in the norm of $Y_1$.

Define the quadratic form

$$\mathcal{F}\,(\zeta,\,\vartheta;\tau) := (\vartheta, F_1 Cw)_\Xi - (\vartheta, F_2\vartheta)_\Xi - \ \tau(G_1 Cw, \xi)_\Xi,$$

$$\zeta = (w, \xi) \in Y_0 \times \Xi,\ \vartheta \in \Xi$$

**(A6)** $\exists\; \tau \in \mathbb{R}, \quad \varkappa\tau \le 0, \quad \exists\; \delta > 0$

$$\mathcal{F}^c\,(\widetilde{\zeta}, \widetilde{\vartheta}; \tau) \le -\delta|\widetilde{\zeta}|^2$$

$$\forall\; \widetilde{\zeta} \in \Xi^c \quad \forall\; \omega \in \mathbb{R}, \quad \widetilde{\zeta} = (i\omega(i\omega\, I - A)^{-1}B\widetilde{\xi}, \widetilde{\xi}), \quad \widetilde{\vartheta} = i\omega\widetilde{\xi}.$$

**(A7)** For any operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ such that

$P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ we have

$$\psi(y_1) - \psi(y_1 - P(y_1 - y_2)) + \psi(y_2) - \psi(y_2 + P(y_1 - y_2)) \ge 0,$$

$$\forall\; y_1, y_2 \in Y_1.$$

On $Y_1$ the function $\psi_p(y) := \psi(y - Py) - \psi(y)$ is convex and lower semi-continuous.

**Theorem** Under the above assumptions any solution $\{y, \xi\}$ of (1),(2) converges in $Y_0 \times \Xi$ to the stationary set $S$ as $t \to \infty$.

## 4. Example

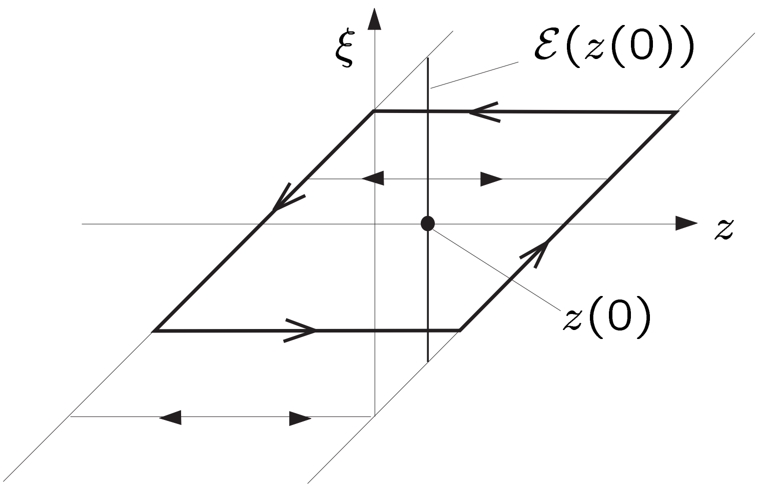
$$
\begin{aligned}
& u_t = u_{xx} - bu\,, \quad x \in (0,1)\,, \quad u(x,0) = u_0(x)\,, \quad b > 0 \\
& u_x(0,t) = 0\ , \quad u_x(1,t) = \rho\,\varphi(z,\xi_0)(t)\ , \quad \rho \in \mathbb{R}\backslash\{0\}\ , \\
& z(t) = \int_0^1 u(x,t)\,dx
\end{aligned} \tag{5}
$$

$\varphi : \mathcal{D}(\varphi) \subset W^{1,1}(0,T) \times \mathbb{R} \to W^{1,1}(0,T)$ play

$(z,\xi_0) \in \mathcal{D}(\varphi) \mapsto \varphi(z,\xi_0)(\cdot)$

$\mathcal{E} : \mathbb{R} \to 2^{\mathbb{R}}$ s. th.

$$\mathcal{D}(\varphi) = \left\{ (z,\xi_0) \in W^{1,1}(0,T) \times \mathbb{R} \;\middle|\; \xi_0 \in \mathcal{E}(z(0)) \right\}$$




**Quadratic constraints**:

**(A1)** : $0 \le \dot\varphi(z,\xi_0)(t)\dot z(t) \le (\dot z(t))^2 \quad \forall\, z \in W^{1,1}(0,T)$

**(A2)** : $\varkappa = -1, \quad \varkappa \int_{t_1}^{t_2} \varphi(z,\xi_0)(t)\dot z(t)dt \ge -\gamma(z(0))$

$$\forall\, z \in W^{1,1}(0,T)\,, \quad \gamma(z(0)) \ge 0$$

**Interpretation as variational equation** $(1'), (2')$:

$Y_0 := L^2(0,1)$ ,

$$(u,v)_0 := \int_0^1 u(x)v(x)dx \ , \quad \forall\, u,v \in L^2(0,1)$$

$Y_1 := W^{1,2}(0,1)$

$$(u,v)_1 := \int_0^1 (u(x)v(x) + u'(x)v'(x))dx, \quad \forall\, u,v \in W^{1,2}(0,1)$$

$A \in \mathcal{L}(Y_1, Y_{-1})$ is given by

$$(Au,v)_{-1,1} := \int_0^1 (Au)(x)v(x)dx =$$

$$- \int_0^1 (u'(x)v'(x) + bu(x)v(x))dx, \quad \forall\, u,v \in W^{1,2}(0,1)$$

$\Xi := \mathbb{R}$ , $B \in \mathcal{L}(\mathbb{R}, Y_{-1})$ is defined by

$$(B\xi, v(x))_{-1,1} = \rho\xi v(1) \ , \quad \forall\, \xi \in \mathbb{R} \ , \quad \forall\, v \in W^{1,2}(0,1)$$

$$\Rightarrow B = [\rho\,\delta(x-1)] \ , \quad \delta\text{-Dirac's distribution}$$

$Z := \mathbb{R}$ , $\quad C \in \mathcal{L}(Y_0, Z)$ is given by

$$Cu := \int_0^1 u(x)dx \ , \quad \forall\, u \in L^2(0,1)$$

$\exists\, \varepsilon > 0 \quad (Au,u)_{-1,1} \le -\|u'(x)\|_0^2 - b\|u\|_0^2 \le -\varepsilon\|u\|_1^2 - b\|u\|_0^2$ ,

$\forall\, u \in W^{1,2}(0,1)$

$\Rightarrow \ (A3), (A4), (A5)$

**Frequency domain condition**:

$\chi(s) = C\widetilde{u}(\cdot, s)$ , $s \in \mathbb{C}$,

transfer function $\widetilde{u}(\cdot, s)$ is the solution of the ordinary BVP

$$\begin{aligned} &(b+s)\widetilde{u} = \widetilde{u}_{xx} \,, \\ &\widetilde{u}_x(0,s) = 0 \,, \quad \widetilde{u}_x(1,s) = 1 \end{aligned} \tag{6}$$

$$\Rightarrow \quad \widetilde{u}(x,s) = \tfrac{\cosh x\sqrt{s+b}}{\sqrt{s+b}\,\sinh\sqrt{s+b}} \,, \quad s \in \mathbb{C}\backslash\{-b\}$$

$$\Rightarrow \quad \chi(s) = \rho \int_0^1 \tilde{u}(x,s)dx = \frac{\rho}{s+b}$$

**(A6)** : $\exists\, \tau \geq 0 \qquad \exists\, \delta > 0 \qquad \forall\, \omega \in \mathbb{R}$ :

$$\mathrm{Re}\,\{\chi(i\omega) + 1 - \tfrac{\tau}{i\omega}\,\chi(i\omega)\} \geq \delta\,|\,\chi(i\omega)\,|^2$$

$$\mathrm{Re}\,\chi(i\omega) = \tfrac{\rho b}{b^2+\omega^2} \,, \quad |\,\chi(i\omega)\,|^2 = \tfrac{\rho^2}{b^2+\omega^2}$$

$$\mathrm{Im}\,\chi(i\omega) = -\tfrac{\rho\,\omega}{b^2+\omega^2}$$

**(A6)** $\Longleftrightarrow$

$\exists\, \tau \geq 0 \qquad \exists\, \delta > 0 \quad \forall\, \omega \in \mathbb{R}$ :

$$\frac{\rho b}{b^2+\omega^2} + 1 + \frac{\tau\rho}{b^2+\omega^2} \geq \frac{\delta\rho^2}{b^2\omega^2}$$

$$\Leftrightarrow \qquad \frac{b}{\rho} + \frac{b^2}{\rho^2} + \frac{\tau}{\rho} > 0\,.$$

**Stationary set of (5)**:

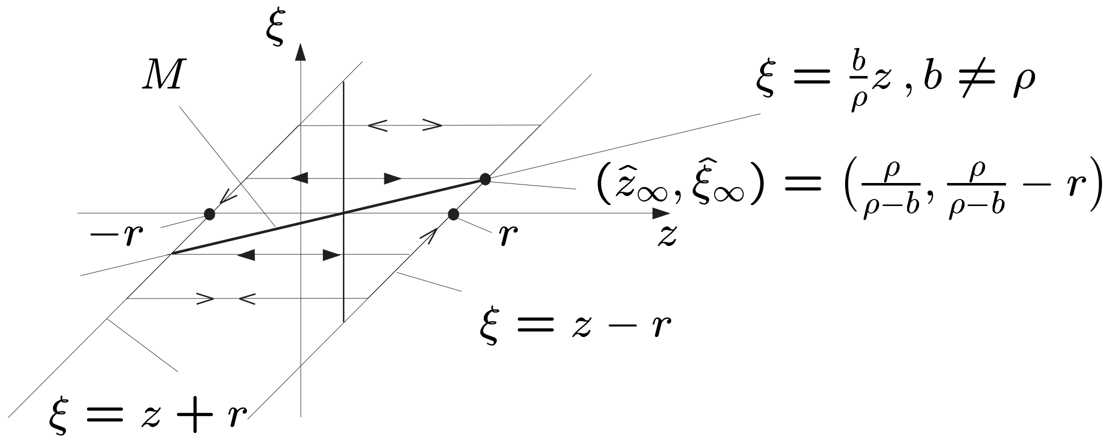
$$0 = \widehat{u}_{xx} - b\widehat{u}\,, \quad x \in (0,1)$$

$$\widehat{u}_x(0) = 0, \quad \widehat{u}_x(1) = \rho\,\varphi(\widehat{z},\widehat{\xi}\,), \quad \widehat{z} = \int_0^1 \widehat{u}(x)dx \qquad (7)$$

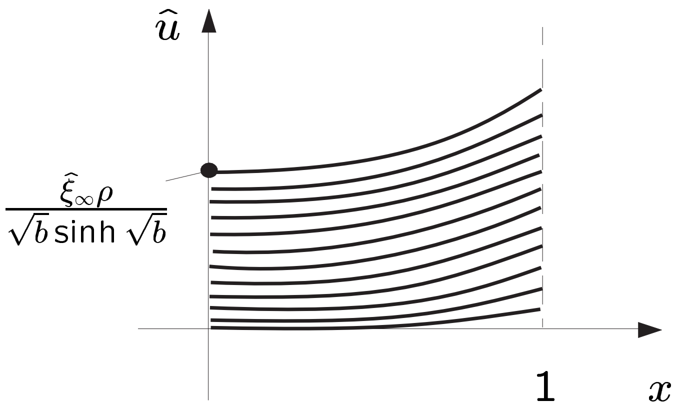
$$\Leftrightarrow \quad \widehat{u}(x) = \tfrac{\cosh\sqrt{b}\,x}{\sqrt{b}\sinh\sqrt{b}}\,\rho\,\varphi(\widehat{z},\widehat{\xi}\,), \quad \widehat{z} = \int_0^1 \widehat{u}(x)dx$$

$$\Leftrightarrow \quad \widehat{z} = \frac{\rho}{b}\varphi(\widehat{z},\widehat{\xi}\,) = \frac{\rho}{b}\widehat{\xi}$$

$$\widehat{u}(x) = \tfrac{\cosh\sqrt{b}\,x}{\sqrt{b}\sinh\sqrt{b}}\,\rho\,\widehat{\xi} \qquad (8)$$




Intersection of $\xi = \frac{b}{\rho}z$ with the graph of the hysteresis




Continuum of stationary temperature fields for the rod

Comparison with the generalized Popov criterion for absolute stability:

$\varphi(z,\xi) = \phi(z) \equiv z$ "nonlinear" function

Frequency-domain condition:

$\exists\, \tau \geq 0 \quad \forall\, \omega \in \mathbb{R} \ : \ 1 + \mathrm{Re}\,[(1 + i\omega\tau)\,\chi(i\omega)] > 0\,.$

$\Leftrightarrow 1 + \frac{\rho\, b}{b^2+\omega^2} + \tau \frac{\rho\, \omega^2}{b^2+\omega^2} > 0$

$\Rightarrow$ absolute stability of (5) as variational system, i.e. each solution $\{y, \xi\}$ belongs to $\mathcal{W}(0,\infty) \times L^2(0,\infty; \Xi)$

$\Rightarrow \ |y(t)|_0 \to 0 \quad$ as $\quad t \to \infty\,,$

$\widehat{y} = 0$ the unique stationary state trajectory.