

**Determining functionals for cocycles
and application to the microwave
heating problem**

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1. Introduction

The notion of a cocycle is a generalization of a dynamical system. Introduce basic notions from the cocycle theory ([P. Kloeden, B. Schmalfuss, 1997]).

Let (Q, d) be a metric space called the base space. The pair $(\{\tau^t\}_{t \in \mathbb{R}}, (Q, d))$ where $\tau^t : Q \rightarrow Q$ for each $t \in \mathbb{R}$ is called the *base flow* if

$$\begin{aligned}\tau^0 &= id_Q, \\ \tau^t \circ \tau^s &= \tau^{t+s} \quad \forall t, s \in \mathbb{R}.\end{aligned}\tag{1}$$

Let (M, ρ) be other metric space called the phase space.

Definition 1 The pair $(\{\varphi^t(q, \cdot)\}_{t \in \mathbb{R}_+, q \in Q}, (M, \rho))$ where $\varphi^t(q, \cdot) : M \rightarrow M$ for each $t \in \mathbb{R}_+, q \in Q$ is called a *cocycle over the base flow* $(\{\tau^t\}_{t \in \mathbb{R}}, (Q, d))$ if

$$\begin{aligned}\varphi^0(q, \cdot) &= id_M \quad \forall q \in Q, \\ \varphi^{t+s}(q, \cdot) &= \varphi^t(\tau^s(q), \varphi^s(q, \cdot)) \quad \forall q \in Q, \quad \forall t, s \in \mathbb{R}_+.\end{aligned}\tag{2}$$

For brevity the cocycle $(\{\varphi^t(q, \cdot)\}_{t \in \mathbb{R}_+, q \in Q}, (M, \rho))$ over the base flow $(\{\tau^t\}_{t \in \mathbb{R}}, (Q, d))$ will be denoted (φ, τ) .

Define the space $W = Q \times M$ and the family of mappings $S^t : W \rightarrow W, t \in \mathbb{R}_+, S^t(q, u) = (\tau^t(q), \varphi^t(q, u))$. The dynamical system $(\{S^t\}_{t \in \mathbb{R}_+}, (W, \tilde{\rho}))$ is called the *skew product*, where

$$\tilde{\rho}((q_1, u_1), (q_2, u_2)) = \max \{ \rho(u_1, u_2), d(q_1, q_2) \}$$

A *nonautonomous set* $\hat{\mathcal{C}} = \{\mathcal{C}(q)\}_{q \in Q}$ is the mapping $Q \rightarrow 2^M$. A nonautonomous set is called *bounded (closed, compact)* if for any $q \in Q$ the set $\mathcal{C}(q)$ is bounded (closed, compact) in M .

A bounded $\hat{\mathcal{C}}$ is called *globally B-pullback absorbing set* for (φ, τ) if for any $q \in Q$ and any bounded set $\mathcal{B} \subset M$ there exists a $T = T(q, \mathcal{B})$ such that $\varphi^t(\tau^{-t}(q, \mathcal{B})) \subset \mathcal{C}(q)$ for $t \geq T$.

A nonautonomous set $\hat{\mathcal{C}}$ is called *globally B-pullback attracting* for (φ, τ) if for any $q \in Q$ and any bounded set $\mathcal{B} \subset M$

$$\lim_{t \rightarrow +\infty} \text{dist}(\varphi^t(\tau^{-t}(q), \mathcal{B}), \mathcal{C}(q)) = 0,$$

where dist is the Hausdorff semidistance in (M, ρ) .

A nonautonomous set $\hat{\mathcal{C}}$ is called *invariant (positively invariant)* for (φ, τ) if for any $q \in Q$ and $t \geq 0$ the equality $\varphi^t(q, \mathcal{C}(q)) = \mathcal{C}(\tau^t(q))$ (inclusion $\varphi^t(q, \mathcal{C}(q)) \subset \mathcal{C}(\tau^t(q))$) holds.

Definition 2 A nonautonomous set is called a *global B-pullback attractor* for the cocycle (φ, τ) if it is compact, invariant and is globally B-pullback attracting.

For the proof of existence of a B-pullback attractor we will use the Kloeden-Schmalfluss theorem:

Theorem 1 Let the cocycle (φ, τ) have a compact globally B-pullback absorbing set $\hat{\mathcal{C}} = \{\mathcal{C}(q)\}_{q \in Q}$.

Then (φ, τ) has a unique B-pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(q)\}_{q \in Q}$, where for each $q \in Q$

$$\mathcal{A}(q) = \bigcap_{t \in \mathbb{R}_+} \overline{\bigcup_{s \geq t, s \in \mathbb{R}_+} \varphi^s(\tau^{-s}(q), \mathcal{C}(\tau^{-s}(q)))}.$$

2. Existence of a B-pullback attractor for the 1-dimensional microwave heating problem

The derivation of 1-dimensional case is given in [H.-M. Yin et al., 2006].

Consider the initial-boundary problem

$$\begin{aligned} \psi_{tt} - \psi_{xx} + \sigma(\theta) \psi_t &= 0, & 0 < x < 1, & t > 0 \\ \theta_t - \theta_{xx} &= \sigma(\theta) \psi_t^2, & 0 < x < 1, & t > 0 \end{aligned} \quad (3)$$

$$\begin{aligned} \psi(0, t) &= f_1(t), \psi(1, t) = f_2(t), & t > 0 \\ \theta(0, t) &= \theta(1, t) = 0, & t > 0 \end{aligned} \quad (4)$$

$$\begin{aligned} \psi(x, 0) &= \psi_0(x), \psi_t(x, 0) = \psi_1(x), & 0 < x < 1 \\ \theta(x, 0) &= \theta_0(x), & 0 < x < 1 \end{aligned} \quad (5)$$

Physical meaning: θ - temperature, ψ - time integral of the nonzero component of electric field, σ - electric conductivity, f_1, f_2 - external perturbations of the electric field.

[H.-M. Yin et al., 2006]: Suppose that

(A1.1) σ is locally Lipschitz on $(0, +\infty)$;

(A1.2) There exist constants $0 < \sigma_0 \leq \sigma_1$ such that $\sigma_0 \leq \sigma(z) \leq \sigma_1$ for any $z > 0$;

(A1.3) σ is monotone decreasing.

(A2) $\psi_0 \in L^2(0, 1), \psi_1 \in L^2(0, 1), \theta_0 \in L^2(0, 1), \theta_0 \geq 0$ a.e. on $(0, 1)$.

(A3) f_1, f_2 are $C^2(\mathbb{R})$ and there exists a constant c such that the functions $|f_1'|, |f_2'|, |f_1''|, |f_2''|$ are bounded on \mathbb{R} by c .

Weak solutions are defined in [H.-M. Yin et al., 2006] by an integral identity.

Modification of the theorem from [H.-M. Yin et al., 2006] for 1-dimensional case:

Theorem 2 *For any $T > 0$ there exists a global weak solution $(\psi(x, t), \theta(x, t))$ of the problem (3)-(5) such that $\psi \in C([0, T]; L^2(0, 1)), \theta \in L^2(0, T; H^1(0, 1)) \cap C([0, T]; L^2(0, 1))$.*

(A4) The weak solution is unique.

Denote $f(x, t) = f_1(t)(1 - x) + f_2(t)x$ and $\Psi(x, t) = \psi(x, t) - f(x, t)$ and get the system with homogeneous boundary conditions, i.e.

$$\begin{aligned} \Psi_t &= \Upsilon - f_t, & 0 < x < 1, & t > 0 \\ \Upsilon_t &= \Psi_{xx} - \sigma(\theta)\Upsilon, & 0 < x < 1, & t > 0 \\ \theta_t - \theta_{xx} &= \sigma(\theta)(\Psi_t + f_t)^2, & 0 < x < 1, & t > 0 \end{aligned} \quad (6)$$

with initial and boundary conditions

$$\Psi(0, t) = \Psi(1, t) = 0, \quad \theta(0, t) = \theta(1, t) = 0, \quad t > 0 \quad (7)$$

$$\begin{aligned} \Psi(x, 0) &= \Psi_0(x) = \psi_0(x) - f(x, 0), & 0 < x < 1 \\ \Upsilon(x, 0) &= \Upsilon_0(x) = \psi_1(x) - f_t(x, 0), & 0 < x < 1 \\ \theta(x, 0) &= \theta_0(x), & 0 < x < 1. \end{aligned} \quad (8)$$

Transformed assumption (A2):

$$\mathbf{(A2')} \quad \Psi_0 \in H_0^1(0, 1), \Upsilon_0 \in L^2(0, 1), \theta_0 \in L^2(0, 1), \theta_0 \geq 0 \text{ a.e. on } (0, 1).$$

Introduction of the cocycle corresponding to the problem (6)-(8). Define the metric space $M = H_0^1(0, 1) \times L^2(0, 1) \times (L^2(0, 1) \cap \{\theta : \theta \geq 0\})$ with the norm

$$\|(\Psi, \Upsilon, \theta)\|_M^2 = \|\Psi_x\|_{L^2(0,1)}^2 + \|\Upsilon\|_{L^2(0,1)}^2 + \|\theta\|_{L^1(0,1)}^2.$$

In our situation: $Q = \mathbb{R}$, $\tau^t(s) = t + s$,

$$\varphi^t(s, u_0) = u(t + s, s, u_0),$$

where $u(t, s, u_0) = (\Psi(\cdot, t), \Upsilon(\cdot, t), \theta(\cdot, t))$ is the solution of (6)-(8) such that $u(s, s, u_0) = u_0 = (\Psi_0, \Upsilon_0, \theta_0)$.

From existence and uniqueness of the solution we conclude ([I. Ermakov, Y. Kalinin, V. Reitman, 2011]):

Theorem 3 *The system (6)-(8) generates a cocycle $(\{\varphi^t(s, \cdot)\}_{t \in \mathbb{R}_+, s \in \mathbb{R}}, (M, \|\cdot\|_M))$ over the base flow $(\{\tau^t\}_{t \in \mathbb{R}}, \mathbb{R})$.*

Proof of the existence of an absorbing set - combination of two techniques:

- Lyapunov function for the 1st block (damped wave equation);
- monotonicity methods for the 2nd block (heat equation).

Damped wave equation. Consider the initial-boundary problem for the wave equation separately:

$$\Psi_{tt} - \Psi_{xx} + \sigma(x, t)\Psi_t = f_{tt} - \sigma(x, t)f_t, \quad 0 < x < 1, \quad t > s \quad (9)$$

$$\Psi(0, t) = \Psi(1, t) = 0, \quad t > s \quad (10)$$

$$\Psi(x, s) = \Psi_0, \quad \Psi_t(x, s) = \Psi_1, \quad 0 < x < 1 \quad (11)$$

where $s \in \mathbb{R}$. Here $\sigma(x, t)$ is a certain function.

Assumptions that are a transformation of (A1)-(A3):

(A1.2*) There exist constants $0 < \sigma_0 \leq \sigma_1$ such that $\sigma_0 \leq \sigma(x, t) \leq \sigma_1$ for all $x \in (0, 1), t \geq 0$.

(A2*) $\Psi_0 \in H_0^1(0, 1), \Psi_1 \in L^2(0, 1)$.

(A3*) The function $f(x, t)$ is C^1 in x , C^2 in t and there exists a constant $c > 0$ such that $|f_t| < c, |f_{xt}| < c, |f_{tt}| < c$ for any $x \in (0, 1), t \in \mathbb{R}$.

Under the above assumptions the problem (9) has a unique solution $(\Psi(\cdot, t), \Psi_t(\cdot, t)) \in M_1 = H_0^1(0, 1) \times L^2(0, 1)$ [R. Temam, 1993]. For $(\Psi, \Upsilon) \in M_1$ define

$$\|(\Psi, \Upsilon)\|_{M_1}^2 = \|\Psi_x\|_{L^2(0,1)}^2 + \|\Upsilon\|_{L^2(0,1)}^2$$

Write the equation (9) as a system

$$\begin{aligned} \Psi_t &= \Upsilon - f_t, \\ \Upsilon_t &= \Psi_{xx} - \sigma(x, t)\Upsilon \end{aligned} \quad (12)$$

with boundary and initial conditions

$$\Psi(0, t) = \Psi(1, t) = 0, \quad t > s \quad (13)$$

$$\Psi(x, s) = \Psi_0(x), \Upsilon(x, s) = \Upsilon_0(x), \quad 0 < x < 1 \quad (14)$$

Proposition 1 *For any $t > 0$ there exist $T > 0$, $c > 0$ such that $\|(\Psi(\cdot, t; s), \Upsilon(\cdot, t; s))\|_{M_1} < c$ for any $s \leq t - T$.*

Idea of the proof: Lyapunov functional on M_1

$$V(\Psi, \Upsilon) = \|\Psi_x\|^2 + 2\lambda(\Psi, \Upsilon) + \|\Upsilon\|^2$$

where $\lambda > 0$ is a parameter. $\|\cdot\|$ and (\cdot, \cdot) are in $L^2(0, 1)$.

Denote $V(t) = V(\Psi(\cdot, t), \Upsilon(\cdot, t))$.

We prove that there exist $\delta > 0$ and $c_1 > 0$ such that

$$\frac{d}{dt}V(t) \leq -\delta V(t) + c_1,$$

$$V(t) \leq e^{-\delta(t-s)}V(s) + c_2,$$

for any t, s , $t \geq s$.

The nonlinear heat equation (2nd equation of (6)).

General setting ([A.A.Pankov, 1983] - boundedness and almost periodicity in time of solutions of variational inequalities):

$E \subset H \subset E'$, where $(E, \|\cdot\|_E)$ is a reflexive Banach space, H is a Hilbert space, E' is the space dual to E , E is continuously and densely embedded into H . $A(t) : E \rightarrow E'$ is a family of monotone operators and $F : \mathbb{R} \rightarrow E'$ is a measurable function. The operator $A(t) : E \rightarrow E'$ is called monotone if

$$(A(t)u - A(t)v, u - v) \geq 0, \quad \forall u, v \in E.$$

(\cdot, \cdot) - duality pairing on $E \times E'$, coinciding on $E \times E$ with the scalar product in H .

Consider the evolution equation

$$\dot{u} + A(t)u = F(t). \quad (15)$$

Suppose that there is an $\alpha > 0$ such that

$$(A(t)u - A(t)v, u - v) \geq \alpha \|u - v\|^2 \quad \forall u, v \in E. \quad (16)$$

Define the following function spaces:

- $C_b(\mathbb{R}, E)$ is the set of continuous functions $f : \mathbb{R} \rightarrow E$, for which $\sup_{t \in \mathbb{R}} \|f(t)\|_E$ is finite and
- $BS^p(\mathbb{R}, E)$, $1 \leq p < \infty$ is the subspace in $L_{loc}^p(\mathbb{R}, E)$, consisting of functions with finite norm

$$\|f\|_{S^p}^p = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|_E^p ds \right).$$

Consider the heat equation in the form

$$\theta_t - \theta_{xx} = \sigma(\theta)g(x, t).$$

Suppose that $g(x, t) \geq 0$ is measurable and uniformly bounded in t . We have $g(x, t) = (\Psi_t(x, t) + f_t(x, t))^2$. For σ the assumptions (A1.1)-(A1.3) hold.

$\sigma(\theta) = \sigma_0 + \tilde{\sigma}(\theta)$, where σ_0 is from (A1.2). We get the initial-boundary problem

$$\theta_t - \theta_{xx} - \tilde{\sigma}(\theta)g(x, t) = \sigma_0g(x, t), \quad 0 < x < 1, \quad t > s \quad (17)$$

$$\theta(0, t) = \theta(1, t) = 0, \quad t > s \quad (18)$$

$$\theta(x, s) = \theta_0(x), \quad 0 < x < 1. \quad (19)$$

The initial-boundary problem (17)-(19) generates an evolution equation (15), where $A(t)u = -u'' - g(x, t)\tilde{\sigma}(u)$ for $u \in E$ and $F(t) = \sigma_0g(\cdot, t)$.

In our situation $E = H_0^1(0, 1)$ and $H = L^2(0, 1)$. Check condition (16). Let $u, v \in H_0^1(0, 1)$, $\zeta = u - v$. Then

$$\begin{aligned} (A(t)u - A(t)v, u - v) &= (-\zeta'', \zeta) + (g(\cdot, t)(\tilde{\sigma}(v) - \tilde{\sigma}(u)), \zeta) = \\ &= (\zeta', \zeta') + (g(\cdot, t)(\tilde{\sigma}(v) - \tilde{\sigma}(u)), \zeta) \geq \|\zeta\|^2. \end{aligned}$$

By [A.A. Pankov, 1983]:

1. The equation (15) has a unique solution $u \in BS^2(\mathbb{R}, H_0^1(0, 1)) \cap C_b(\mathbb{R}, L^2(0, 1))$. For the equation (17) this means that there exists a constant c_1 such that $\|\theta(\cdot, t; s)\| \leq c_1$ for any $t, s \in \mathbb{R}$, $s \leq t$.

2. We have the estimate

$$\|\theta_1(\cdot, t; s) - \theta_2(\cdot, t; s)\| \leq e^{-c_2(t-s)} \|\theta_{01} - \theta_{02}\|, \quad (20)$$

where $\theta_i(x, t; s)$ is the solution of (17) with initial data θ_{0i} and initial time s .

The constant c_1 does not depend on initial data:

$$\theta(x, t; s) = \int_0^1 G(x, y; t, s) \theta_0(y) dy + \int_s^t \int_0^1 G(x, y; t, r) g(y, r) dr dy,$$

where $G(x, y; t, r)$ is the corresponding Green's function,

$$\|G(x, y; t, s)\| \leq \frac{c_3}{\sqrt{t-s}}.$$

The influence of initial data tends to zero.

We can make the initial time s tend to $-\infty$, which corresponds to time shift in $g(x, t)$.

Proposition 2 *Let $\theta(\cdot, t; s)$ be the solution of (17)-(19). There exists a constant c such that for all t and $s \leq t$ the inequality $\|\theta(\cdot, t; s)\| \leq c$ holds where c does not depend on θ_0 .*

From uniform boundedness in s of solutions of the wave equation and the heat equation we obtain

Theorem 4 *The cocycle (φ, τ) generated by problem (6) – (8) has a globally B -pullback absorbing set.*

Applying the Kloeden-SchmalFUSS Theorem, we get

Theorem 5 (*I. Ermakov, Y. Kalinin, V. Reitmann, 2011*) *The cocycle (φ, τ) generated by problem (6) – (8) has a global B-pullback attractor.*

3. Determining functionals for cocycles

The concept of determining functionals for evolution equations was introduced in [I.D. Chueshov, 1998]. Physical meaning: asymptotically finite-dimensional dynamics.

If the system has a global attractor, such functionals can give an approximation of the attractor.

Let $(\{S^t\}_{t \in \mathbb{R}_+}, (E, \|\cdot\|))$ be a dynamical system on Banach space $(E, \|\cdot\|)$.

Definition 3 *The set $\{l_j\}_{j=1}^N$ of linear continuous functionals on E is called a set of asymptotically determining functionals for the dynamical system $(\{S^t\}_{t \in \mathbb{R}_+}, (E, \|\cdot\|))$ if for any $u_1, u_2 \in E$ the condition*

$$\lim_{t \rightarrow +\infty} |l_j(S^t(u_1)) - l_j(S^t(u_2))| = 0, \quad j = 1, \dots, N$$

implies

$$\lim_{t \rightarrow +\infty} \|S^t(u_1) - S^t(u_2)\| = 0.$$

Introduce the determining modes which are an important case of determining functionals.

Definition 4 *The determining modes for the dynamical system $(\{S^t\}_{t \in \mathbb{R}_+}, (H, (\cdot, \cdot)))$ on a Hilbert phase*

space $(H, (\cdot, \cdot))$ are determining functionals $l_j(\cdot) = (\cdot, e_j)$ where $\{e_j\}_1^N$ are some elements of H .

The notion of pullback- asymptotically determining functionals for processes was introduced in [J.A. Langa, 2003]. We give a generalization for cocycles.

Next we present the results from [I. Ermakov, Y. Kalinin, V. Reitmann, 2011].

Definition 5 The set $\{l_j\}_{j=1}^N$ of linear continuous functionals on Banach space $(M, \|\cdot\|)$ is called a set of pullback- asymptotically determining functionals for the cocycle $(\{\varphi^t(q, \cdot)\}_{q \in Q, t \in \mathbb{R}_+}, (M, \|\cdot\|))$ over the base flow $(\{\tau^t\}_{t \in \mathbb{R}}, (Q, d))$ if the condition

$$\lim_{t \rightarrow +\infty} |l_j(\varphi^t(\tau^{-t}(q), u_1)) - l_j(\varphi^t(\tau^{-t}(q), u_2))| = 0$$

for any $q \in Q$, $u_1, u_2 \in M$, $j = 1, \dots, N$ implies

$$\lim_{t \rightarrow +\infty} \|\varphi^t(\tau^{-t}(q), u_1) - \varphi^t(\tau^{-t}(q), u_2)\| = 0.$$

Let (φ, τ) be a cocycle with a Hilbert phase space H , π_1 is a projector from H onto a finite-dimensional subspace of H , π_2 is its complement. Suppose that

(H1) The non-autonomous set $\{\mathcal{C}(q)\}_{q \in Q}$ is positively invariant for (φ, τ) .

(H2) For any $q \in Q$ there exists $\delta = \delta(q) \in (0, 1)$ such that for all $s \geq 1$, $u, v \in \mathcal{C}(\tau^{-s}(q))$

$$\|(\pi_2)(\varphi^1(\tau^{-s}(q), u) - \varphi^1(\tau^{-s}(q), v))\| \leq \delta(q) \|u - v\|.$$

Let $a_1, a_2 : Q \rightarrow H$ be such mappings that $a_i(q) \in \mathcal{C}(q)$ for any $q \in Q$.

(H3) For any $\varepsilon > 0$, $t \geq 0$ there exists an $L = L(\varepsilon) \in \mathbb{N}$ such that for any $q \in Q$

$$\delta(q)^{2L} \left\| \varphi^{t-L}(q, a_1(q)) - \varphi^{t-L}(q, a_2(q)) \right\|^2 < \varepsilon,$$

and $L(\varepsilon) \rightarrow \infty$ if $\varepsilon \rightarrow 0$.

The next theorem is a generalization of Theorem 14 in [J.A. Langa, 2003]

Theorem 6 *Let the assumptions (H1)-(H3) hold and there exists a $\beta > 0$ such that for any $q \in Q$*

$$\lim_{t \rightarrow +\infty} \left\| \pi_1(\varphi^t(\tau^{-t}(q), a_1(q)) - \varphi^t(\tau^{-t}(q), a_2(q))) \right\| \leq \beta.$$

Then

$$\lim_{t \rightarrow +\infty} \left\| \varphi^t(\tau^{-t}(q), a_1(q)) - \varphi^t(\tau^{-t}(q), a_2(q)) \right\| \leq \beta. \quad (21)$$

Corollary 1 *Let there exist a $\beta > 0$ such that for all $q \in Q, u, v \in H$*

$$\lim_{t \rightarrow +\infty} \left\| \pi_1(\varphi^t(\tau^{-t}(q), u) - \varphi^t(\tau^{-t}(q), v)) \right\| \leq \beta.$$

Then

$$\lim_{t \rightarrow +\infty} \left\| \varphi^t(\tau^{-t}(q), u) - \varphi^t(\tau^{-t}(q), v) \right\| \leq \beta.$$

This corollary gives the existence of pullback- asymptotically determining modes for a cocycle.

Now consider cocycles of a special type. Such cocycles are generated by the microwave heating problem.

Let (φ, τ) be a cocycle with phase space $E = E_1 \times E_2$ where E_1 is a Hilbert and E_2 is a Banach space, respectively.

φ has the form (φ_1, φ_2) , i.e.

$$\varphi_1 : \mathbb{R}_+ \times Q \times E_1 \times E_2 \rightarrow E_1,$$

$$\varphi_2 : \mathbb{R}_+ \times Q \times E_1 \times E_2 \rightarrow E_2.$$

Let π_1 be the projector from E_1 onto a finite-dimensional subspace of E_1 , π_2 be its orthogonal complement.

Let $a_1, a_2 : Q \rightarrow E$ be such mappings that $a_i(q) \in \mathcal{C}(q)$ for any $q \in Q$.

Modify assumptions (H2) and (H3) so that they hold for φ_1 instead of φ .

(H2*) For any $q \in Q$ there exists $\delta = \delta(q) \in (0, 1)$ such that for all $s \geq 1, u, v \in \mathcal{C}(\tau^{-s}(q))$

$$\|\pi_2(\varphi_1^1(\tau^{-s}(q), u) - \varphi_1^1(\tau^{-s}(q), v))\|_{E_1} \leq \delta(q) \|u - v\|_E.$$

(H3*) For any $\varepsilon > 0, t \geq 0$ there exists an $L = L(\varepsilon) \in \mathbb{N}$ such that for any $q \in Q$

$$\delta(q)^{2L} \|\varphi_1^{t-L}(q, a_1(q)) - \varphi_1^{t-L}(q, a_2(q))\|_{E_1}^2 < \varepsilon,$$

and $L(\varepsilon) \rightarrow \infty$ if $\varepsilon \rightarrow 0$.

Theorem 7 *Suppose that*

1) *The assumptions (H1),(H2*),(H3*) hold for the cocycle (φ, τ) .*

2) *The estimate*

$$\|\varphi_2^t(\tau^{-t}(q), u_1, u_2) - \varphi_2^t(\tau^{-t}(q), v_1, v_2)\|_{E_2} \leq e^{-ct} \|u_2 - v_2\|_{E_2}$$

holds with some constant $c > 0$ for any $t > 0$, $u_1, v_1 \in E_1$, $u_2, v_2 \in E_2$.

3) *There exists a $\beta > 0$ such that for any $q \in Q$*

$$\lim_{t \rightarrow +\infty} \|\pi_1(\varphi_1^t(\tau^{-t}(q), a_1(q)) - \varphi_1^t(\tau^{-t}(q), a_2(q)))\|_{E_1} \leq \beta.$$

Then

$$\lim_{t \rightarrow +\infty} \|\varphi^t(\tau^{-t}(q), a_1(q)) - \varphi^t(\tau^{-t}(q), a_2(q))\|_E \leq \beta. \quad (22)$$

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