Reconstructing attractors of infinite-dimensional dynamical systems from low-dimensional projections

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1 Feedback control systems

Suppose

$$\dot{y} = f(y) \tag{1.1}$$

with a vector function $f : \mathbb{R}^n \to \mathbb{R}^n$ (*"parent flow"*) is given.
Then (1.1) can be written as *feedback control system*

$$\dot{y} = Ay + B\phi\left(Cy(t)\right) , \tag{1.2}$$

where A, B and C are arbitrary $n \times n$ matrices (B and C regular) and $\phi(\sigma) = B^{-1}[f(C^{-1}\sigma) - AC^{-1}\sigma], \sigma \in \mathbb{R}^n$. Consider the more general system

$$\dot{y} = Ay + B\xi(t) \ , \ \xi(t) = \phi(Cy(t), \xi_0) \tag{1.3}$$

with the $n \times n, n \times m$ and $l \times m$ matrices A, B and C and the nonlinearity ϕ which can be smooth, piecewise smooth or a hysteresis function.

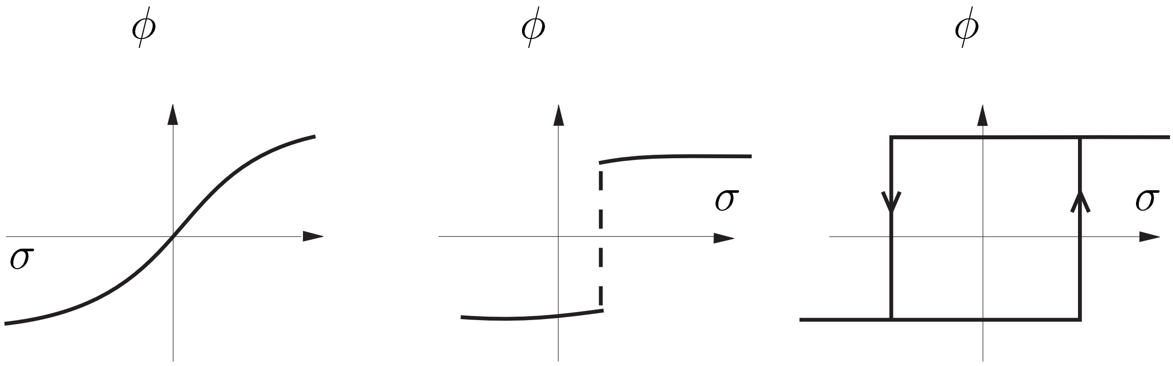

Fig. 1

Example 1.1 dry friction, elasto-plastic deformation (Fig. 1) □

Remark 1.1 (1.3) can also describe an infinite-dimensional system. Suppose $Y_1 \subset Y_0 \subset Y_{-1}$ are densely and continuously embedded Hilbert spaces (*rigged Hilbert space structure*), Z and W are also Hilbert spaces,

$$A : Y_1 \to Y_{-1} \ , \quad B : \Xi \to Y_{-1} \ , \quad C : Y_1 \to W$$

are bounded linear operators, $\phi : W \to \Xi$ is a nonlinearity, and the equation

$$\dot{y} = Ay + B\phi\,(Cy) \tag{1.4}$$

is the *state space realization model* for well-posed input-output (measurement) maps.

- ODE case: $Y_1 = Y_0 = Y_{-1} = \mathbb{R}^n$, $W = \mathbb{R}^s$, $\quad \Xi = \mathbb{R}^r$
- PDE (Boundary control system)

$Y_0 = L^2(0,1),\ Y_1 = W^{1,2}(0,1),\ Y_{-1} = Y^*,\ A : Y_1 \to Y_{-1}$,

$(Au,v)_{1,-1} = \int_0^1 (Au)(x)v(x)dx = -\int_0^1 (au_x v_x + bu\,v)dx$,

$\forall\, u, v \in W^{1,2}(0,1)$

$\Xi = \mathbb{R},\ B : \Xi \to Y_{-1},\ B = a\delta(x-1),\ g : \mathbb{R} \to \mathbb{R},\ a > 0, b > 0$ numbers

$$\left.\begin{array}{l} \frac{\partial u}{\partial t} = au_{xx} - bu,\ 0 < x < 1\ , \\ u_x(0,t) = 0,\ u_x(1,t) = g(w(t))\ ,\ u(\cdot,0) = u_0 \\ g(w(t)) = Cu(x,t) = \int_0^1 c(x)u(x,t)dx,\ c \in L^2(0,1)\ . \end{array}\right\} \tag{1.5}$$

- Functional differential equations (FDE's or PDE's with delay)

$$\dot{y}(t) = \sum_{k=0}^{m} A_k y(t + r_k) + B\phi(Cy_t)\,, -r \le r_m < \cdots < r_1 < r_0 = 0\,, \tag{1.6}$$

$y(0) = h \in H,\ y_0 = \alpha \in L^2([-r,0];H), H$ Hilbert space

$y_t(\cdot) : [-r,0] \to H,\ y_t(\Theta) = y(t+\Theta)$ a.a. $\Theta \in [-r,0]$

$A_i : \mathcal{D}(A_i) \subset H \to H,\ i = 0,1,\ldots,m\,,\ Y_0 = L^2([-r,0];H) \times H$,

$B \in \mathcal{L}\,(U,H)$, U Hilbert space

$F : \mathcal{D}(F) \subset Y_0 \to Y_0$ given by $F(\{\alpha,h\}) := \{\dot{\alpha}, \sum_{k=0}^m A_k h(r_k) + B\phi(C\alpha)\}$

$\mathcal{D}(F) = \{\ \{\alpha,h\} \in Y_0 \,|\, \alpha : [-r,0] \to H$ absolutely continuous,
$\dot{\alpha} \in L^2([-r,0];H),\ h = \alpha(0) \in \mathcal{D}(A)\}$ ODE in the *skew-product* Y_0

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}\bar{\phi}(\bar{C}z(t)) \equiv F(z(t)),\ z(0) = z_0 \in Y_0 \tag{1.7}$$

$(\{\alpha,h\},\{\beta,k\})_0 := \int_{-r}^0 (\alpha(\Theta),\beta(\Theta))_H\, d\Theta + (h,k)_H$

for $\{\alpha,h\},\ \{\beta,k\} \in Y_0$

$H = \mathbb{R}^n : \dot{y} = \int_{-r}^0 \Gamma(s)y(t+s)ds + A_1 y(t) + A_2 y(t-r) + b\varphi(\sigma(t))$,

$\sigma(t) = c^* y(t) + \int_{-r}^0 g^*(s)y(t+s)ds, \quad y(0) = h, y_0 = \alpha\,$,

with b and c n-vectors, $g \in L^2([-r,0];\mathbb{R}^n)$,

$\Gamma \in L^2([-r,0];\ \mathbb{R}^{n\times n})$, A_1 and A_2 $n \times n$ matrices,

$\varphi : \mathbb{R} \to \mathbb{R}$ s.t. the generalized solutions exist

Some solution conceptions for (1.3)

1) Weak solutions in some Sobolev space
2) Classical solutions for differential inclusions
3) Filippov solutions, i.e. absolutely continuous functions $y(\cdot)$ which satisfy (1.3) almost everywhere.

(**H1**) For any initial state (1.3) has exactly one Filippov solution on $[0, \infty)$.

2 The reconstruction principle and the cone condition

Let $\gamma = \{y(t) | t \geq 0\}$ be a semi-orbit of (1.3), Π the projection on some plane E (Fig. 2).

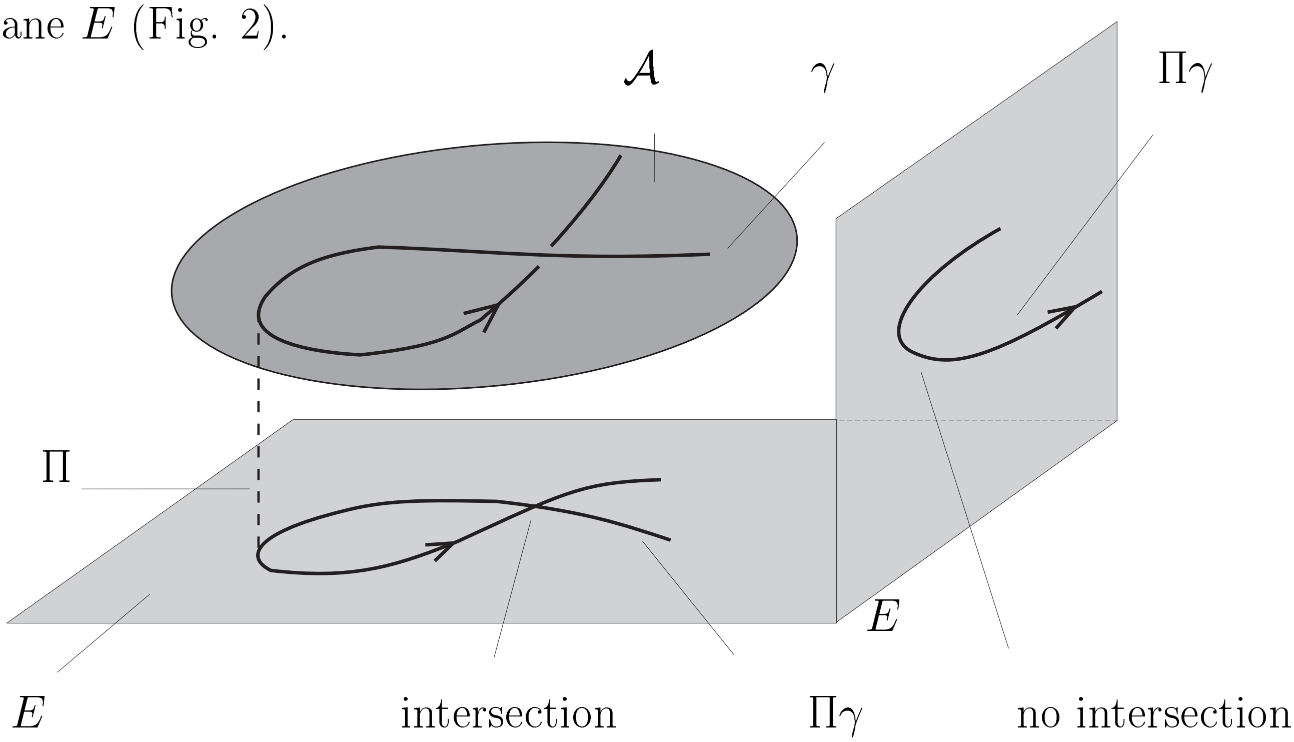

Fig. 2

How to choose a projection $\Pi : \mathbb{R}^3 \to E \cong \mathbb{R}^2$ such that $\Pi : \gamma \to \Pi\gamma$ is one-to-one and continuous in $\mathcal{A}$?

(**H2**) (*cone condition*) There exist a set $S \subset \mathbb{R}^n$ and an $n \times n$-matrix $P = P^*$ having 2 negative and $(n-2)$ positive eigenvalues such that for any two solutions $y_1(\cdot), y_2(\cdot)$ of (1.3) with $y_i(t) \in S, \forall\, t \geq 0, i = 1, 2,$ we have with $V(y) = y^* P y$ the inequality

$$V(y_1(t) - y_2(t)) \leq 0\,, \quad \forall\, t \geq 0 \tag{2.1}$$

[10] Smith, [2] Foias et al, [7] Robinson.

Geometrical interpretation of the cone condition for $\mathbf{n=3}$

Assume $V(y) = y^*Py$ is a quadratic form satisfying (2.1) along the solutions of (1.3), $K := \{y|V(y) \le 0\}$ is a 2-dimensional cone, $\mathbb{R}^3\backslash K$ is a 1-dimensional cone (Fig.3). Let l be the direction of the main axis of $\mathbb{R}^3\backslash K$ with $l^*Pl > 0, E$ is the orthogonal to l plane through the origin, Π is the orthogonal projection on E.
Suppose that $y_1(\cdot), y_2(\cdot)$ are two arbitrary distinct solutions of (1.3) in S, i.e. $y_1(t) \ne y_2(t) \quad \forall\, t \ge 0, y_1(t), y_2(t) \in S, \quad \forall\, t \ge 0$. From (2.1) we have $V(y_1(t) - y_2(t)) \le 0,\ \forall\, t \ge 0$, i.e. $y_1(t) - y_2(t) \in K, \quad \forall\, t \ge 0$.
Then

$$\Pi\, y_1(t) \ne \Pi\, y_2(t), \quad \forall\, t \ge 0\ . \tag{2.2}$$

Assume the opposite, i.e. assume that

$$\exists\, t_0 \ge 0 : \Pi\, y_1(t_0) = \Pi\, y_2(t_0)\ . \tag{2.3}$$

It follows from (2.3) that $\Pi\,[y_1(t_0) - y_2(t_0)] = 0$, i.e. the point $y_1(t_0) - y_2(t_0)$ is projected under Π into 0. But then there exists a $k \ne 0$ such that $y_1(t_0) - y_2(t_0) = kl$. Consequently we have $V(kl) = k^2 l^*Pl > 0$, a contradiction to the fact that $V(y_1(t_0) - y_2(t_0)) \le 0$.

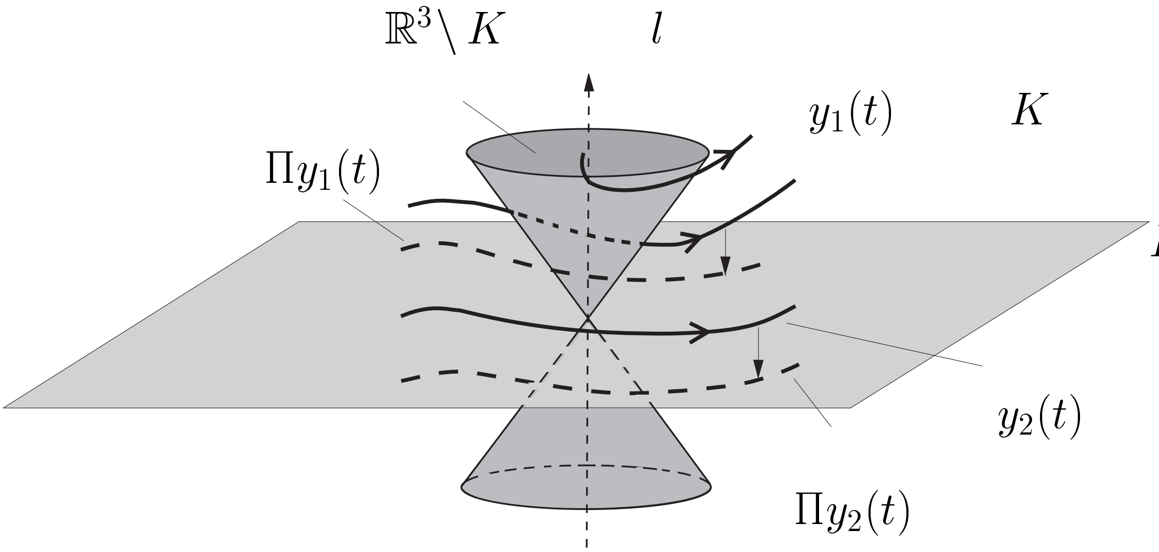

Fig. 3

3 Frequency-domain methods

Suppose A, B and C are matrices of order $n \times n, n \times m$ and $l \times n$, respectively, $F(x, \xi)$ is a *Hermitian form* on $\mathbb{C}^n \times \mathbb{C}^m$, i.e. a quadratic form which takes only real values. The pair (A, B) is called *stabilizable* if there exists an $n \times m$ matrix D such that $A + BD$ is Hurwitzian, i.e. has only eigenvalues with negative real part.

Theorem 3.1 *(Frequency theorem; Yakubovich, 1962; Kalman, 1963) Let the pair (A, B) be stabilizable and* $\det(i\omega I - A) \neq 0,\ \forall\, \omega \in \mathbb{R}$.

a) For the existence of a real symmetric $n \times n$-matrix P satisfying the Riccati inequality

$$\begin{aligned} 2\,\mathrm{Re}\, x^* P(Ax + B\xi) + F(x, \xi) < 0, \\ \forall\, x \in \mathbb{C}^n \quad \forall\, \xi \in \mathbb{C}^m, |x| + |\xi| \neq 0 \end{aligned} \tag{3.1}$$

it is necessary and sufficient that the frequency-domain condition

$$\begin{aligned} F((i\omega I - A)^{-1} B\,\xi, \xi) < 0, \\ \forall\, \xi \in \mathbb{C}^m, \xi \neq 0 \quad \forall\, \omega \in \mathbb{R} \end{aligned} \tag{3.2}$$

is satisfied.

b) A matrix $P = P^*$ satisfying (3.1) can be computed in a finite number of steps.

Consider the system

$$\dot{y} = Ay + B\phi(Cy(t))\ , \tag{3.3}$$

where A, B and C are matrices of order $n \times n, n \times 1$ and $1 \times n$, respectively. Introduce the *transfer function* $\chi(z) = C(zI - A)^{-1}B$ for $z \in \mathbb{C} : \det(zI - A) \neq 0$.

$\phi : \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

(**H3**) There exist parameters $\mu_1 < 0 < \mu_2$ such that

$$\mu_1(\sigma_1 - \sigma_2)^2 \leq [\phi(\sigma_1) - \phi(\sigma_2)](\sigma_1 - \sigma_2) \leq \mu_2(\sigma_1 - \sigma_2)^2 \quad \forall\, \sigma_1, \sigma_2 \in \mathbb{R} \tag{3.4}$$

Remark 3.1 If ϕ is C^1 the condition (3.4) can be written in the following way:

(**H3**)' There exist parameters $\mu_1 < 0 < \mu_2$ such that

$$\mu_1 \leq \phi'(\sigma) \leq \mu_2, \quad \forall\, \sigma \in \mathbb{R} \qquad (3.4)'$$

□

Theorem 3.2 *Suppose that for ϕ from (3.3) the condition* (**H3**) *is satisfied and there exists a $\lambda > 0$ such that the following holds:*
1) The pair $(A + \lambda I, B)$ is stabilizable ;
2) The matrix $A + \lambda I$ has exactly two eigenvalues with positive real part and $(n-2)$ with negative real part;
3) $\mathrm{Re}\,[1+\mu_1\chi(i\omega-\lambda)]\,[1+\mu_2\chi(i\omega-\lambda)]^* > 0,\ \forall\, \omega \in \mathbb{R}$;

(Gap condition)

Then there exists an $n \times n$-matrix $P = P^*$ having 2 *negative and* $(n-2)$ *positive eigenvalues, and a number $\varepsilon > 0$ such that with the function $V(y) = y^* P y$ the inequality*

$$\frac{d}{dt}V(y_1(t) - y_2(t)) + \lambda V(y_1(t) - y_2(t)) - \varepsilon|y_1(t) - y_2(t)|^2,\ \forall\, t \geq 0 \tag{3.5}$$

(Squeezing property)

is satisfied for any two solutions $y_1(\cdot), y_2(\cdot)$ of (3.3).

Proof of Theorem 3.2 Suppose $y_1(\cdot), y_2(\cdot)$ are two arbitrary solutions of (3.3). Then $y := y_1 - y_2$ is a solution of

$$\dot{y} = Ay + B\psi \text{ with } \psi(t) := \phi(\sigma_1(t)) - \phi(\sigma_2(t)),$$

$\sigma_i(t) := Cy_i(t), i = 1, 2.$
By assumption (**H3**) we have with $\sigma = \sigma_1 - \sigma_2$ the inequality

$$\mu_1\sigma(t)^2 \leq \psi(t)\sigma(t) \leq \mu_2\sigma(t)^2,\ \forall\, t \geq 0\ . \tag{3.6}$$

Because of 1) and 3) Theorem 3.1 is applicable with the Hermitian form $F(y,\xi) = \mathrm{Re}\,[(\mu_2 Cy - \xi)(\xi - \mu_1 Cy)^*]$ (Fig. 4). It follows that there exist an $n \times n$-matrix $P = P^*$ and a number $\varepsilon > 0$ such that

$$2\,y^* P[(A+\lambda I)y + B\psi] + (\mu_2 Cy - \psi)(\psi - \mu_1 Cy) \le -\,\varepsilon\big[\,|y|^2 + |\psi|^2\,\big] \quad \forall\, y \in \mathbb{R}^n,\ \forall\, \psi \in \mathbb{R}\,. \tag{3.7}$$

For $\psi = 0$ we get from (3.7) the inequality

$$2\,y^* P(A+\lambda I)y - \mu_1\mu_2 (Cy)^2 \le -\varepsilon |y|^2,\ \forall\, y \in \mathbb{R}^n\,. \tag{3.8}$$

Since $\mu_1\mu_2 < 0$ inequality (3.8) implies that

$$y^* P(A+\lambda I)y + y^*(A+\lambda I)^* P y < 0,\ \forall\, y \in \mathbb{R}^n \quad y \ne 0\,. \tag{3.9}$$

From (3.9) it follows by Lyapunov's theorem that the matrix P has exactly 2 negative and $(n-2)$ positive eigenvalues, since $A + \lambda I$ has 2 eigenvalues with positive real part and $(n-2)$ eigenvalues with negative real part.
Putting in (3.7) $y = y_1 - y_2,\ \psi = \phi(Cy_1) - \phi(Cy_2)$ and using the fact that

$$[\mu_2 C(y_1 - y_2) - (\phi(Cy_1) - \phi(Cy_2))]\,[(\phi(Cy_1) - \phi(Cy_2)) - \mu_1 C(y_1 - y_2)] \ge 0\,,$$

we derive from (3.7) the inequality

$$\frac{d}{dt}V(y_1(t) - y_2(t)) + 2\lambda V(y_1(t) - y_2(t)) \le -\varepsilon |y_1(t) - y_2(t)|^2,\ \forall\, t \ge 0\,. \qquad \blacksquare$$

Geometrical interpretation of the frequency-domain condition

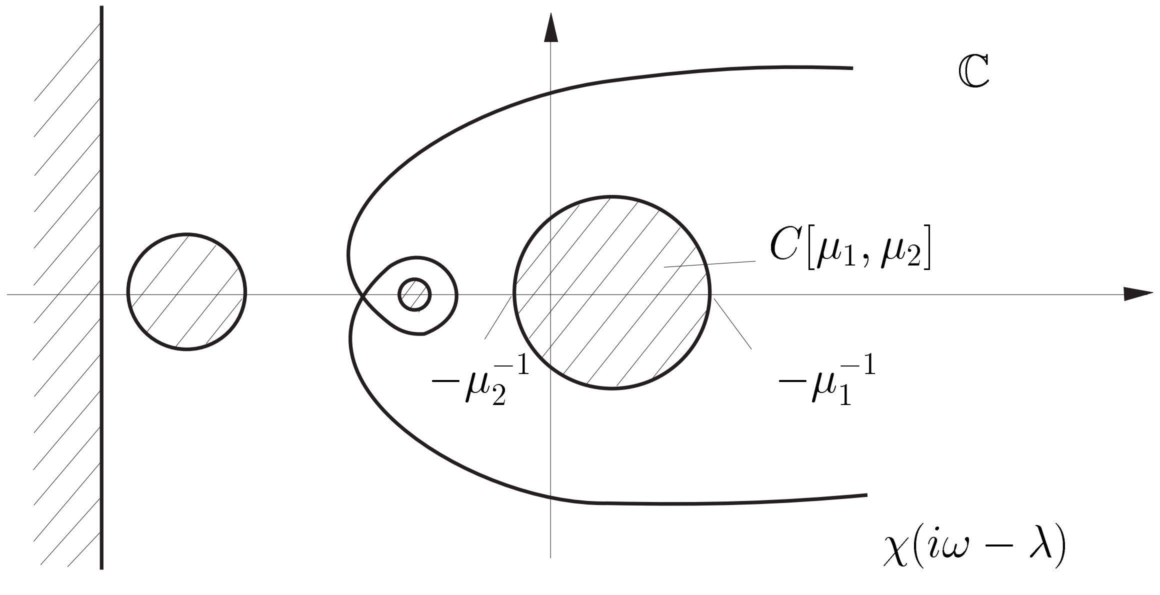

Fig. 4

4 Amenable solutions and essential modes

Definition 4.1 *(R. A. Smith, 1987) Suppose $\lambda > 0$ is a number. A solution $y(\cdot)$ of (1.3) is called* amenable *if there exists a number $\tau \in \mathbb{R}$ such that $y(t) \in S$, $\forall\, t \le \tau$, and $\int_{-\infty}^{\tau} e^{2\lambda t} |y(t)|^2 dt < +\infty$.*

Remark 4.1 If (1.3) has a compact attractor then all solutions inside the attractor are amenable. □

Theorem 4.1 *Suppose that the conditions of Theorem 3.2 are satisfied with a parameter* $\lambda > 0$ *and* $P = P^*$ *is the* $n \times n$ *matrix satisfying (3.7) and having 2 negative and* $(n-2)$ *positive eigenvalues.*
Choose a matrix $Q = Q^*$ *of order* $n \times n$ *such that*

$$Q^* P Q = \begin{pmatrix} -1 & & & & \\ & -1 & & & 0 \\ & & +1 & & \\ 0 & & & \ddots & \\ & & & & +1 \end{pmatrix}$$

and define the linear map $\Pi : \mathbb{R}^n \to \mathbb{R}^2$ *by* $\Pi\, y := u$ *where* $\binom{u}{v} = Q^{-1} y$ *with* $u \in \mathbb{R}^2$,
$v \in \mathbb{R}^{n-2}$. *Then if* $\mathcal{A}$ *is the set of amenable solution of (3.3) the map*

$$\Pi : \mathcal{A} \to \Pi\, \mathcal{A} \tag{4.1}$$

is a homeomorphism, i.e. one-to-one and bicontinuous.

Definition 4.2 *(O. Ladyzhenskaya [5]) Suppose that (1.4) has in the (infinite-dimensional) phase-space* Y_0 *an attractor* $\mathcal{A}$ *and a finite-dimensional projector* Π *with the following property: For any two orbits* γ_1, γ_2 *of the attractor* $\mathcal{A}$ *the condition* $\Pi\, \gamma_1 = \Pi\, \gamma_2$ *implies* $\gamma_1 = \gamma_2$. *Then we say that the* number of essential or determining modes of (1.4) for $\mathcal{A}$ is finite.

Corollary 4.1 *Suppose that the conditions of Theorem 3.2 are satisfied and (3.3) has a compact attractor* $\mathcal{A}$. *Then the number of essential modes for* $\mathcal{A}$ *is two.*

Remark 4.2 In many cases in the system $\dot{y} = Ay + B\phi(Cy)$ (1.4) we have a symmetric $A = A^* : Y_1 \to Y_{-1}$. If the embedding $Y_1 \subset Y_{-1}$ is completely continuous then the operator A has a system of eigenfunctions (modes) $\{w_j\}$ associated to eigenvalues $\{\lambda_j\}$ by $Aw_j = \lambda_j w_j, w_j \in Y_1, \lambda_i < \lambda_{i+1}, \lambda_i \to +\infty, (w_j, w_k) = \delta_j^k$ such that $\{w_j\}$ is a basis of Y_1, i.e. any element y can be written as $y = \sum y_j w_j, \sum y_j^2 < \infty$.
Then $\Pi\, y := (y_1, y_2) \in \mathbb{R}^2$ or, more general, $\Pi\, y = (y_1, \ldots, y_i) \in \mathbb{R}^i$ is a finite-dimensional projection. Physically this means that the *total energy of an orbit is dominated by the energy of the first i modes.* □

Proof of Theorem 4.1 (See also Smith [10]) $\frac{d}{dt}[e^{2\lambda t}V(y_1 - y_2)] \leq -2\,\varepsilon e^{2\lambda t}|y_1 - y_2|^2$, $\forall\, t \leq \tau$, if $y_1, y_2 \in S$. Integration on $[\Theta, \tau]$ gives

$$e^{2\lambda\tau}V(y_1(\tau) - y_2(\tau)) \leq e^{2\lambda\Theta}V(y_1(\Theta) - y_2(\Theta)) - \; 2\,\varepsilon \int_{\Theta}^{\tau} e^{2\lambda t}|y_1(t) - y_2(t)|^2 dt. \tag{4.2}$$

Since $e^{\lambda t}|y_1(t)|, e^{\lambda t}|y_2(t)|$ are in $L^2(-\infty, \tau)$ the function $e^{\lambda t}|y_1 - y_2|$ is also in $L^2(-\infty, \tau)$.
It follows that there exists a sequence of times $\Theta_\nu \to -\infty$ as $\nu \to \infty$ with $|y_1(\Theta_\nu) - y_2(\Theta_\nu)|e^{\lambda\Theta_\nu} \to 0$. Putting in (4.2) $\Theta = \Theta_\nu$ and assuming $\nu \to \infty$ we get

$$e^{2\lambda\tau}V(y_1(\tau) - y_2(\tau)) \leq -2\,\varepsilon \int_{-\infty}^{\tau} e^{2\lambda t}|y_1(t) - y_2(t)|^2 dt \leq 0\;. \tag{4.3}$$

Take a regular $n \times n$-matrix $Q = Q^*$ such that

$$Q^*PQ = \begin{pmatrix} -1 & & & & \\ & -1 & & 0 & \\ & & +1 & & \\ 0 & & & \ddots & \\ & & & & +1 \end{pmatrix}$$ and put $y = Q\binom{u}{v}$ with $u \in \mathbb{R}^2, v \in \mathbb{R}^{n-2}$,

$\Pi\, y := u, \forall\, y \in \mathbb{R}^n$. Clearly that $|\Pi\, y|^2 = |u|^2$. Since $Q^{-1}y = \binom{u}{v}$ we have $|Q^{-1}y|^2 = |u|^2 + |v|^2$ and$V(y) = y^*Py = (u^*, v^*)Q^*PQ\binom{u}{v} = -|u|^2 + |v|^2$.
It follows that

$$\begin{aligned} V(y) + 2|\Pi\, y|^2 &= -|u|^2 + |v|^2 + 2|u|^2 = |u|^2 + |v|^2 \\ &= \; |Q^{-1}y|^2 \geq |\Pi\, y|^2\,, \quad \forall\, y \in \mathbb{R}^n\;. \end{aligned}$$

Consider two arbitrary amenable solutions y_1, y_2 of (4.3). It follows now that $V(y_1(t) - y_2(t)) \leq 0$, $\forall\, t \geq 0$, and

$$2\,|\Pi\,(y_1(\tau) - y_2(\tau))|^2 \geq |Q^{-1}(y_1(\tau) - y_2(\tau))|^2 \geq |\Pi\,(y_1(\tau) - y_2(\tau))|^2\;. \tag{4.4}$$

If h and k are arbitrary constants the amenable solutions $y_1(t-h), y_2(t-k)$ can replace y_1, y_2 in (4.4). Thus, if γ_1, γ_2 are amenable orbits of y_1, y_2 then

$$\boxed{2\,|\Pi\, p_1 - \Pi\, p_2|^2 \geq |Q^{-1}(p_1 - p_2)|^2 \geq |\Pi\, p_1 - \Pi\, p_2|^2 \qquad \forall\, p_1, p_2 \in \gamma_1, \gamma_2.} \tag{4.5}$$

It follows now that $\Pi : \mathcal{A} \to \Pi\, \mathcal{A}$ is a homeomorphism of $\mathcal{A}$ onto $\Pi\, \mathcal{A}$. ■

5 Lipschitz manifolds and the extension procedure

Consider (3.3) under the assumptions of Theorem 4.1 and let

$$h : \Pi\,\mathcal{A} \to \mathcal{A} \tag{5.1}$$

be the inverse map of $\Pi\ : \mathcal{A} \to \Pi\,\mathcal{A}$, (4.1), where $\mathcal{A}$ is again the set of amenable solutions.
It follows from (4.5) that

$$2\,|u_1 - u_2|^2 \geq |Q^{-1}(h(u_1) - h(u_2))|^2 \geq |u_1 - u_2|\ , \qquad \forall\, u_1, u_2 \in \Pi\,\mathcal{A}\ . \tag{5.2}$$

If $y(\cdot)$ is an amenable solution of (3.3) then $u(t) := \Pi\, y(t)$ is the solution of the
2-dimensional *reduced* or *observation ODE*

$$\dot{u} = \underbrace{\Pi\, f(h(u))}_{=:g(u)} \qquad (f(y) = Ay + B\phi(Cy)). \tag{5.3}$$

The reduced vector field g is defined only on the closed set $\Pi\,\mathcal{A} \subset E \cong \mathbb{R}^2$, since h is defined only on $\Pi\,\mathcal{A}$. Can we extend h to a Lipschitz continuous map

$$\tilde{h} : E \cong \mathbb{R}^2 \to \mathbb{R}^n\ (Y_0)\ ?$$

Assume for a moment that this is possible. Then it holds:

1) $\tilde{g} := \Pi\,(f(\tilde{h}))$ is a Lipschitz vector field on $E \cong \mathbb{R}^2$ if f is Lipschitz : $\tilde{g} = \Pi\ \circ f \circ \tilde{h}$.
It follows that all solutions of (3.2) exist and are unique. The *observation ODE (5.2) can be used for the reconstruction of the set $\mathcal{A}$ of (3.3).*

2) The set $\mathcal{A}$ of amenable solutions of (3.3) lies in the set

$$\mathcal{M} := \{y \in \underset{(Y_0)}{\mathbb{R}^n} | y = \tilde{H}(u),\ u \in \underset{(\mathbb{R}^m)}{\mathbb{R}^2}\}\ . \tag{5.4}$$

Since $\tilde{h}$ is Lipschitz the set (5.4) is a 2-dimensional (m-dimensional) Lipschitz manifold. If $\mathcal{A}$ is the global attractor the set $\mathcal{M}$ attracts all orbits of (3.3) from $\mathbb{R}^n(Y_0)$. In this case $\mathcal{M}$ is called the *inertial manifold* of (3.3) ([2] Foias et al; [7] Robinson) .

Theorem 5.1 (Stein's extension theorem [11])
Let X be a closed subset of $\mathbb{R}^m$, $H(= Y_0)$ be a Hilbert space, and $h : X \to H$ be a continuous function.
Then there is a continuous extension $\tilde{h} : \mathbb{R}^m \to H$ and there exists a $K = K(m)$ such that if $|h(x) - h(y)| \leq C|x - y|, \forall\, x, y \in X$, then $|\tilde{h}(x) - \tilde{h}(y) \leq KC|x - y|, \forall\, x, y \in \mathbb{R}^m$.

Corollary 5.1 *Under the conditions of Theorem 4.1 the reduced vector field (5.2) can be extended to a Lipschitz vector field in $E \cong \mathbb{R}^2$. Any amenable solution y of the infinite-dimensional vector field*
$\dot{y} = Ay + B\phi$ in the phase space Y_0 can be represented as $y = \tilde{h}(u(t))$, where $u(t)$ is the unique solution of the reduced equation (5.2) with initial state $u(0) = \Pi\, y(0)$.

6 Constructing a reduced system from measurements

Suppose

$$\dot{y} = f(y) \tag{6.1}$$

is a given (unknown) dissipative system in $\mathbb{R}^n$ with attractor $\mathcal{A}$.

Step 1: **Choice of the linear part**
Choose a number $\lambda > 0$ and matrices A, B and C of order $n \times n, n \times 1$ and $1 \times n$, respectively, such that $(A + \lambda I, B)$ is stabilizable, and $A + \lambda I$ has $2(m)$ eigenvalues with positive real part and $n - 2$ eigenvalues with negative real part.

Step 2: **Reconstruction of the class of nonlinearities**
Calculate on $[0, T]$ the linear semigroup $S(t) = e^{At}$ with A from Step 1. Take an $\varepsilon < 0$ (tolerance), a natural number N and observe near the attractor the solutions $y_i(\cdot), i = 1, 2, \ldots, N$, of (6.1) on $[0, T]$. Find for any $i = 1, 2, \ldots, N$ a solution $\phi_i \in L^\infty(0, T; \mathbb{R}^n)$ of the linear inequality

$$\sup_{t\in[0,T]} |y_i(t) - S(t)y_i(0) - \int_0^t S(t-s)B\phi_i(s)ds| < \varepsilon\, . \tag{6.2}$$

It follows that $\phi_i(t) \approx \phi(Cy_i(t))$ in the sense of $L^2(0, T)$, where
$\dot{y}_i(t) = Ay_i + B\phi(Cy_i(t))$ on $[0, T]$.

Determine two constants $-\infty \le \mu_1 < \mu_2 \le +\infty$ ($\mu_2 < +\infty$ if $\mu_1 = -\infty$ and $\mu_1 > -\infty$ if $\mu_2 = +\infty$) such that

$$\mu_1 [C(y_i(t) - y_j(t))]^2 \le [\phi_i(t) - \phi_j(t)]\, C\, [y_i(t) - y_j(t)] \\ \le \ \mu_2 [C(y_i(t) - y_j(t))]^2 , \quad i,j = 1,\dots,N \quad t \in [0,T] . \qquad (6.3)$$

Take two constants $-\infty \le \mu_1 < \mu_2 \le +\infty$ such that

$$\mu_1 [C(y_i(t) - y_j(t))]^2 \le [\phi_i(t) - \phi_j(t)]\, C\, [y_i(t) - y_j(t)] \\ \le \ \mu_2 [C(y_i(t) - y_j(t))]^2,\ i,j = 1,\dots,N,\ t \in [0,T] .$$

Step 3: **Graphic test of the frequency-domain / gap condition**
Compute the frequency-domain characteristic
$\chi(i\omega - \lambda) = C((i\omega - \lambda)I - A)^{-1}B$ and compare with the circle $C[\mu_1, \mu_2]$ with $\mu_1 < \mu_2$ from Step 2 (Fig. 5).

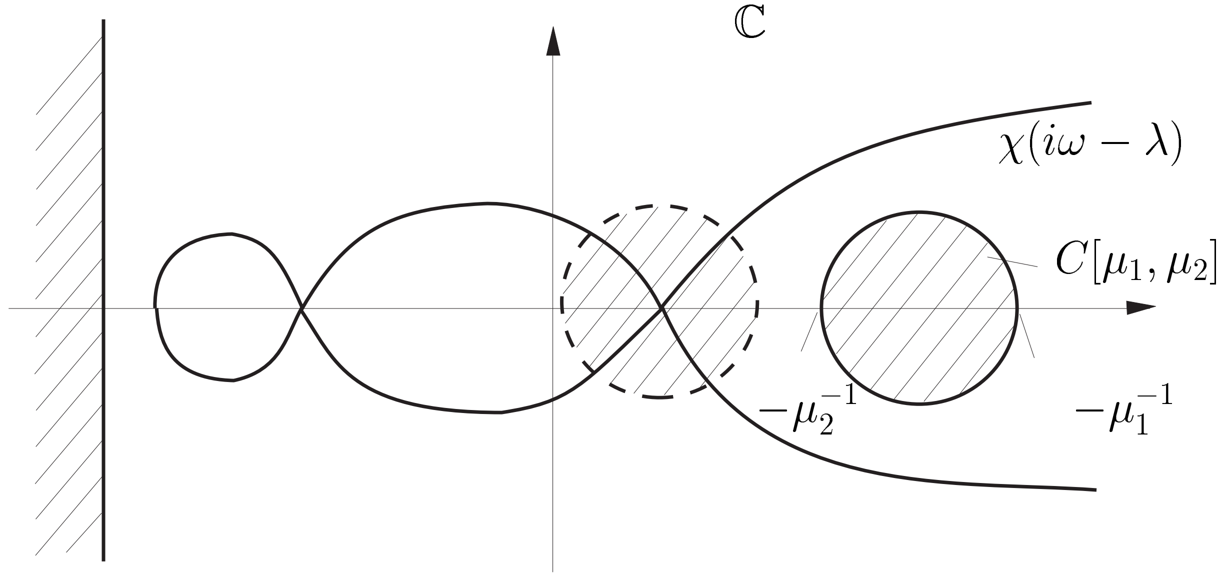

Fig. 5

If there is no intersection between $\chi(i\omega - \lambda)$ and $C[\mu_1, \mu_2]$ go to Step 4. In other case change A, B, C or m and begin again with Step 1.

Step 4: **Calculation of a homeomorphism $\Pi : \mathcal{A} \to \Pi\,\mathcal{A}$**
Find with A, B, C from Step 1 and $\mu_1 < \mu_2$ from Step 3 an $n \times n$ matrix $P = P^*$ of the matrix inequality

$$2\, y^* P\, [(A + AI)y + B\psi] + (\mu_2 Cy - \psi)\,(\psi - \mu_1 Cy) < 0 , \\ \forall\, y \in \mathbb{R}^n,\ \forall\, \psi \in \mathbb{R}, |y| + |\psi| \ne 0. \qquad (6.4)$$

Such a solution exists by the frequency theorem and is computable in a finite number of steps. Any solution $P = P^*$ of (6.3) has 2 negative and $n-2$ positive eigenvalues. Define a matrix $Q = Q^*$ through

$Q^*PQ = \begin{pmatrix} -1 & & & & \\ & -1 & & & 0 \\ & & +1 & & \\ 0 & & & \ddots & \\ & & & & +1 \end{pmatrix}$. Then the projection is $\Pi : \mathbb{R}^n \to \mathbb{R}^2$

defined by $\Pi\, y = u, y \in \mathbb{R}^n, u \in \mathbb{R}^2, v \in \mathbb{R}^{n-2}$, s.th. $\binom{u}{v} = Q^{-1}y$.
It follows from Theorem 4.1 that of $\mathcal{A}$ is the amenable set of (6.1) then $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ is a homeomorphism.

Step 5: **Determination of a reduced ODE for the full equation**
Let $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ be the homeomorphism from Step 4. Determine a reduced 2-dimensional ODE $\dot{u} = \underbrace{\Pi\, f(\tilde{h}(u))}_{\tilde{g}(u)}$ from the observations $\Pi\, y_i(t)$, where $y_i(t)$ are arbitrary solutions of (6.1) near the attractor and use constructively the extension theorem of Stein to extend this vector field from the closed set $\Pi\mathcal{A} \subset E \cong \mathbb{R}^2$ to a Lipschitz vector field on the whole E.

7 When is a given linear projection a homeomorphism on the attractor?

Suppose

$$\dot{y} = f(y) \tag{7.1}$$

is on ODE in $\mathbb{R}^n$. $\mathcal{A}$ is the set of amenable solutions and $\Pi : \mathbb{R}^n \to \mathbb{R}^k$ is a given linear projection. Under what conditions is $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ a homeomorphism?
Write (7.1) again in the form

$$\dot{y} = Ay + B\phi\,(\Pi y)\ , \tag{7.2}$$

where A and B are $n \times n$ and $n \times m$ matrices, and $B\phi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $B\phi\,(\Pi y) := f(y) - Ay$. Assume that $f(0) = 0$ and the solutions of (7.1) exist on $\mathbb{R}_+$ and are unique. Let $K \subset \mathbb{R}^n$ be an invariant and absorbing cone for (7.2) having the property

$$K \cap \{y \in \mathbb{R}^n \mid \Pi y = 0\} = \{0\} \ . \tag{7.3}$$

If (7.3) is satisfied then $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ is a homeomorphism.

(**H3**)" There exists a $k \times m$ matrix M such that

$$0 \leq (\Pi(y_1 - y_2))^* M[\phi(\Pi y_1) - \phi(\Pi y_2)] \ , \quad \forall y_1, y_2 \in \mathbb{R}^n \ .$$

Define the Hermitian form $F_{\mathbb{C}}(y, \xi) := \mathrm{Re}\,(y^*\Pi^* M\xi), y \in \mathbb{C}^n, \xi \in \mathbb{C}^m$, and the transfer matrix $\chi(i\omega) := (i\omega I - A)^{-1}B$.

Theorem 7.1 *Suppose that* (**H3**)" *is satisfied and there exists a* $\delta > 0$ *such that the following holds:*

1) The pair $(A + \lambda I, B)$ *is stabilizable ;*

2) The matrix $A + \lambda I$ *has* k *eigenvalues with positive real part and* $n - k$ *with negative real part ;*

3) $\mathrm{Re}\, F_{\mathbb{C}}(\chi(i\omega - \lambda)\xi, \xi) < 0, \quad \forall\, \xi \in \mathbb{C}^m, \xi \neq 0, \ \forall\, \omega \in \mathbb{R}$;

4) $\xi^* B^* \Pi^* M \xi \geq 0 \ , \quad \forall\, \xi \in \mathbb{R}^m$.

Then there exists a symmetric $n \times n$ *matrix* P *having* k *negative and* $n - k$ *positive eigenvalues such that the following holds:*

a) The k*-dimensional cone* $K := \{y \in \mathbb{R}^n | y^* P y \leq 0\}$ *is positively invariant for all solutions of (7.1) ;*

b) $K \cap \{y \in \mathbb{R}^n | \Pi y = 0\} = \{0\}$ *;*

c) K *absorbs* $\mathcal{A}$ *and, consequently,* $\Pi : \mathcal{A} \to \Pi\mathcal{A} \subset \mathbb{R}^k$ *is a homeomorphism .*

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