1 The contact-impact problem in continuum mechanics

1.1 Basic facts from finite-deformation theory

Deformation is a one-parametric family of smooth maps $\{\boldsymbol{\Phi}^t\}_{t\in[0,T]}$

$$\boldsymbol{\Phi}^t : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \,.$$

In local coordinates

$$x^i = x^i\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t\right) \quad , \quad i = 1,2,3 \quad , \quad t \in [0,T]\,.$$

The deformation tensor $\boldsymbol{F}^t$ in the point $\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3\right)$ with respect to the basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ is defined by

$$\boldsymbol{F}^t = \frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \,.$$

Here $\{e_i \wedge e_j\}_{1\le i<j\le 3}$ denotes the basis of the second exterior power $(\mathbb{R}^3)^{\wedge 2}$.
Suppose $S \subset \Omega$ a small surface with outer normal $\boldsymbol{n}(\boldsymbol{x})$ in the point $\boldsymbol{x} \in S$ and S' the surface after deformation under $\boldsymbol{F}^t$
The force acting on a small part dS' from the positive normal side is defined by $s^{ij}(\boldsymbol{x})n_i dS$, where n_i are the components of $\boldsymbol{n}(\boldsymbol{x})$
s^{ij} is the first Piola-Kirchhoff tensor
If the columns of $\left(\frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\right)$ are linearly independent we can write

$$s^{ij} = \sigma^{il}\frac{\partial x^j}{\partial \overset{\circ}{x}{}^l} \,.$$

σ^{il} is the second Piola-Kirchhoff tensor

$$x^i = \overset{\circ}{x}{}^i + u^i(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t)$$

Small strain tensor e_{ij}

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \overset{\circ}{x}{}^j} + \frac{\partial u_j}{\partial \overset{\circ}{x}{}^i}\right)$$

Lagrange strain tensor

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2}\frac{\partial u^k}{\partial \overset{\circ}{x}{}^i}\frac{\partial u_k}{\partial \overset{\circ}{x}{}^j}$$

Equilibrium equation with body forces f^j and material density ρ

$$\frac{\partial \sigma^{\alpha j}}{\partial x^\alpha} + \rho\left(f^j - \frac{\partial^2 u^j}{\partial t^2}\right) = 0$$

<u>In tensor notation</u>:
The Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l}\right) g^{lk}$$

Contravariant differentiation procedure

$$u_{j,i} \equiv \nabla_i u_j = \frac{\partial u_j}{\partial \xi^i} - \Gamma^k_{ij} u_k$$
$$u^j_{\ ,i} \equiv \nabla_i u^j = \frac{\partial u^j}{\partial \xi^i} + \Gamma^j_{ki} u^k \,.$$

Small strain and finite strain tensors

$$e_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$$

$$\left.\begin{array}{rcl} d\overset{\circ}{s}{}^2 & = & \overset{\circ}{g}_{ij}\ d\xi^i\ d\xi^j \\ ds^2 & = & g_{ij}\ d\xi^i\ d\xi^j \end{array}\right\} \Rightarrow ds^2 - d\overset{\circ}{s}{}^2 = 2\varepsilon_{ij}\ d\xi^i\ d\xi^j$$

Lagrange strain tensor

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \overset{\circ}{g}_{ij}) = e_{ij} + \frac{1}{2} g^{\alpha\beta} \nabla_i u_\alpha \nabla_j u_\beta \,.$$

Newton's law of motion $[\sigma^{ij}(\delta^l_j + u^l_{,j})]_{,i} = \rho \ddot{u}^l$

1.2 Constitutive law

Plasticity domain K on the body
$K := \{\sigma^{ij} : \mathcal{H}(\sigma^{ij}) \leq 0\}$

a) *von Mises material:* $\mathcal{H}(\sigma^{ij}) = \frac{1}{2} s^{ij} s_{ij} - k^2$
$s^{ij} = s_{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ as deviator of σ^{ij} and $k \neq 0$ is a constant.

b) *Tresca material:* $\mathcal{H}(\sigma^{ij}) = \max|\sigma_i - \sigma_j| - k$
where the maximum is computed over all eigenvalues of the tensor σ^{ij} and $k > 0$ is again a constant.

Total strain is the sum of an elastic part and a plastic part, i.e.

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}.$$

Deformation theory

$$\varepsilon^p_{ij} = \frac{1}{2\mu} \frac{\sqrt{\frac{1}{2}\, s_{ij} s^{ij}} - k}{\sqrt{\frac{1}{2}\, s_{ij} s^{ij}}}$$

where s_{ij} is the deviator stress and μ is a material constant.
Flow theory

$$\dot{\varepsilon}^p_{ij} = \lambda \frac{\partial Y}{\partial \sigma^{ij}} \qquad\qquad \dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}.$$

1.3 Plasticity zone in the spinning disc and in the annular disc

Circular disc of thickness h and density ρ rotating with constant angular velocity ω.
(r, φ, z) are the cylindrical coordinates and τ_r and τ_φ are the non-zero Cauchy stress components

$$\frac{\partial \tau_r}{\partial r} + \frac{\tau_r - \tau_\varphi}{r} = -\rho\,\omega^2 r\,.$$

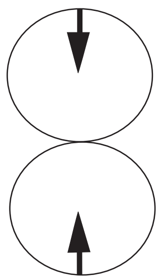
Example (ODE from the theory of plasticity)
$\dot{x}_1(t) = g(t) - x_2(t) \qquad , \; g(t) \quad$ given force
$$\dot{x}_2(t) = \begin{cases} 0 \quad \text{if} \quad |x_2(t)| = 1 \quad \text{and} \quad x_1(t)x_2(t) \geq 0 \\ \beta x_1(t) \qquad \text{otherwise} \end{cases}$$

$\Rightarrow$ a) ODE with discontinuous right-hand side (Filippov's theory)
J. L. Buhite, D.R. Owen
Arch. Rational. Mech.Anal.
61, 357 - 383, 1981

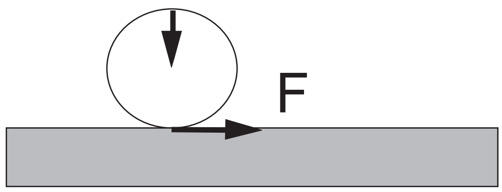
b) Differential inclusions

$$\dot{x}(t) + Ax(t) \ni f(t)$$
(maximal monotone operators in the sense of Brezis)

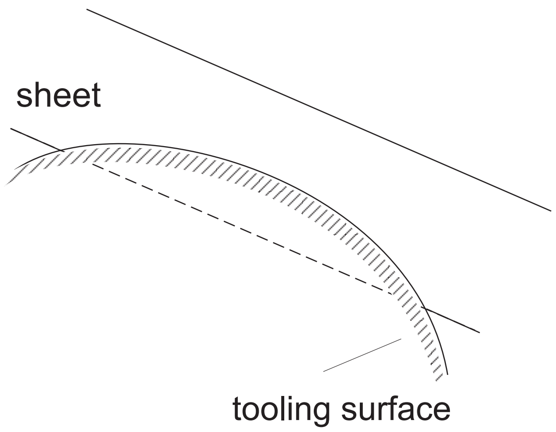
Types of contact



Two circular discs in frictionless contact



A long roller
and a thick elastic body
in frictional contact



Elastic-plastic contact between tooling surface and a sheet

Von Mises-plasticity law possessing the yield condition

$$\tau_r^2 - \tau_r \tau_\varphi + \tau_\varphi^2 = k^2 \qquad (k = \text{const})\,.$$

Connection between Cauchy stresses and small strains e_r and e_φ

$$\begin{aligned} e_r &= \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\left(\tau_r - \tau\right), \\ e_\varphi &= \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\left(\tau_\varphi - \tau\right). \end{aligned}$$

New variables

$$\rho = \frac{r}{r_0} \quad , \quad \tau_\rho = \frac{\tau_r}{k} \quad \text{and} \quad \tau_\varphi := \frac{\tau_\varphi}{k}$$

and the constants

$$a = \frac{r_*}{r_0} \quad , \quad \alpha = \frac{E}{k} \quad \text{and} \quad \lambda = \frac{\rho\,\omega^2 r_0^2}{k}\,.$$

$$\frac{\partial \tau_\rho}{\partial \rho} + \frac{\tau_\rho - \tau_\varphi}{\rho} = -\lambda\rho \quad , \quad 0 \le \rho \le 1\,,$$

$$\tau_\rho^2 - \tau_\rho \tau \varphi + \tau_\varphi^2 = 1\,.$$

Ansatz (Arutyunyan et al., 1987)

$$\begin{aligned} \tau_\rho &= \frac{2}{\sqrt{3}}\cos\left(\Phi + \frac{\pi}{6}\right)\,, \\ \tau_\varphi &= \frac{2}{\sqrt{3}}\cos\left(\Phi + \frac{\pi}{6}\right) \end{aligned}$$

ODE problem

$$\rho\,\frac{d\,\Phi}{d\rho} = \frac{\lambda\rho^2\frac{\sqrt{3}}{2} - \sin\Phi}{\sin(\Phi + \frac{\pi}{6})}$$

Compatibility condition

$$\frac{\partial\, e_\rho}{\partial \rho} + \frac{e_\varphi - e_\rho}{\rho} = 0\,,$$

Linear ODE problem

$$\frac{1+\nu}{2\alpha\sqrt{3}}\left(\sqrt{3}\sin\omega + \cos\omega\right)\frac{d\Psi}{d\rho} + \Psi\left[\frac{1+\nu}{\alpha\rho}\sin\omega + \right.$$
$$\left. + \frac{\sqrt{3}}{2}\frac{d\Psi}{d\rho}\left(\sqrt{3}\cos\omega - \sin\omega\right)\right] = \frac{1-2\nu}{\sqrt{3}\alpha}\sin\omega\frac{d\Psi}{d\rho}$$

Elastic domain $a \le \rho \le 1$.
Hooke's law

$$\begin{aligned} e_\rho &= \frac{1}{2}\left(\tau_\rho - \nu\tau_\varphi\right), \\ e_\varphi &= \frac{1}{2}\left(\tau_\varphi - \nu\tau_\rho\right). \end{aligned}$$

General solution

$$\tau_\rho = C\Big(1 - \frac{1}{\rho^2}\Big) + \frac{\lambda(3+\nu)}{8}\,(1-\rho^2)\,,$$
$$\tau_\rho = C\Big(1 + \frac{1}{\rho^2}\Big) + \frac{\lambda}{8}\,[\,3+\nu-(1+3\,\nu)\rho^2]\,.$$

For $\rho = a(\rho)$ we have to guarantee the continuity of the radial and tangential stresses

$$C\Big(1-\tfrac{1}{a^2}\Big) + \tfrac{\lambda(3+\nu)}{8}(1-a^2) = \tfrac{2}{\sqrt{3}}\cos\Big(\Phi(a)+\tfrac{\pi}{6}\Big)$$

$$C\Big(1+\tfrac{1}{a^2}\Big) + \tfrac{\lambda}{8}\,[\,3+\nu-(1+3\nu)a^2] = \tfrac{2}{\sqrt{3}}\cos\Big(\Phi(a)-\tfrac{\pi}{6}\Big)$$

$$\tau_\rho = C\Big(1 - \frac{e_\varphi^2}{u_\rho^2}\Big) + \frac{\lambda(3+\nu)}{8}\Big(1-\frac{u_\rho^2}{e_\varphi^2}\Big)\,,$$
$$\tau_\rho = C\Big(1 + \frac{e_\varphi^2}{u_\rho^2}\Big) + \frac{\lambda}{8}\,\Big[\,3+\nu-(1+3\nu)\frac{u_\rho^2}{e_\varphi^2}\,\Big]\,.$$

Annular plate of thickness h, of outer radius b and of inner radius a, clamped at the inner edge with the outer edge free subjected to uniform radial compression p at the inner edges

For the exactness of the deformation theory it is in the following assumed that (Korovlev, 1971)

$$\frac{a}{b} \geq 0.37$$

Lamé formula for the tangential and radial stresses depending on the actual radius r by (Filin, 1975)

$$\tau_r = \frac{p\,a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)$$

and

$$\tau_\varphi = \frac{p\,a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)\,.$$

The stress intensity is

$$\tau_{\rm int} = \frac{3\beta}{4}\,(\tau_\varphi - \tau_r)$$

with $\beta = \frac{2+\sqrt{3}}{2}$.

a) Elastic zone:
$$\tau_r^e = -\frac{2E}{3}A\left(\frac{1}{r^2}-\frac{1}{b^2}\right)\,,$$
$$\tau_\varphi^e = -\frac{2E}{3}A\left(\frac{1}{r^2}+\frac{1}{b^2}\right)\,,$$

b) Plastic zone:
$$\tau_r^p = \frac{4k}{3\beta}\left(\ln\frac{r}{a}-\kappa\right)\,,$$
$$\tau_\varphi^p = \frac{4k}{3\beta}\left(\ln\frac{r}{a}-\kappa+1\right)$$

with $\kappa = \frac{3p\beta}{4k}$.
The continuity and compatibility conditions lead to

$$\tau^e_{r_{|r=r_o}} = \tau^p_{r_{|r=r_o}}$$

and

$$\tau^e_\varphi - \tau^e_r = \frac{4k}{3\beta} .$$

Critical pressure

$$p_{\text{cr}} = \frac{4k}{3p} \ln \frac{a}{b} .$$

$a_s := \frac{b}{a}$ is the spinning ratio in metal forming process.

1.4 Friction theory

Displacement $\boldsymbol{u}$ tangential and normal parts

$$\boldsymbol{u} = \boldsymbol{u}_T + u_N \boldsymbol{n} ,$$

where $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_T = (\text{id} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{u}$. Surface stress $\boldsymbol{p}$

$$\boldsymbol{p} = \boldsymbol{p}_T + p_N \boldsymbol{n} ,$$

p_N is the contact pressure.
The tangential relative velocity $\dot{\boldsymbol{u}}_T$ is decomposed into the adherence part and the slipping part

$$\dot{\boldsymbol{u}}_T = \dot{\boldsymbol{u}}^{ad}_T + \dot{\boldsymbol{u}}^{sl}_T .$$

Adherence part

$$\boldsymbol{p}_T = -k\boldsymbol{u}^{ad}_T$$

where k is the elastic contact stiffness.
Slipping part

$$\dot{\boldsymbol{u}}^{sl}_T = -\dot{\gamma} \frac{\partial \Psi}{\partial \boldsymbol{p}_T} \quad ,$$

where the slip potential Ψ determines the direction of slip, and γ is a real function. The state of friction is determined by the slip function Φ and the loading and unloading conditions

$$\begin{array}{lll} \text{loading:} & \Phi > 0 \quad , & \dot{\gamma} = 0 \, , \\ \text{unloading:} & \Phi = 0 \quad , & \dot{\gamma} > 0 \end{array}$$

and

$$\begin{array}{ll} \Phi \leq 0 \Rightarrow & \text{adherence,} \\ \Phi > 0 \Rightarrow & \text{slipping.} \end{array}$$

(Ronda and Colville, 1995)

1.5 Large deformation dynamic elastic-plastic contact problem

${}^t\Omega = {}^t\Omega^A \cup {}^t\Omega^B, \;\; {}^t\Gamma = \partial\, {}^t\Omega = \partial\, {}^t\Omega^A \cup \partial\, {}^t\Omega^B = {}^t\Gamma^A \cup {}^t\Gamma^B$

(i) Equilibrium equations

$$\left(\sigma^{kl}\delta^i_l + \sigma^{kl}u^i_{,l}\right)_{,k} + \rho f^i = \rho f^i_{\mathcal{I}} + \rho f^i_C + \rho f^i_Z, \quad t \in (0,T), \quad \boldsymbol{x} \in {}^t\Omega\,.$$

(ii) Kinematic boundary conditions (prescribed displacements)

$$u_i(\boldsymbol{x},t) = U_i(\boldsymbol{x},t), \quad \boldsymbol{x} \in {}^t\Gamma_U = {}^t\Gamma^A_U \cup {}^t\Gamma^B_U,\; t \in (0,T)\,.$$

(iii) Prescribed boundary forces

$$[\sigma^{kl}\delta^i_l + \sigma^{kl}u^i_{,l}]n_k = F^i,$$

$(n_1, n_2, n_3$ components of $\boldsymbol{n})$

$$\boldsymbol{x} \in {}^t\Gamma^B_F, \quad t \in (0,T)\,.$$

(iv) Tangential frictional stress

$$\sigma^{ij}n_j - \sigma^{jk}n_j n_k n^i = \mathcal{F}^i, \quad \boldsymbol{x} \in {}^t\Gamma_{\mathcal{F}}, \quad t \in (0,T)\,.$$

(v) Initial conditions

$$u_i(\boldsymbol{x},0) = U_{0i}(\boldsymbol{x}), \qquad \dot{u}_i(x,0) = U_{1i}(\boldsymbol{x})\;.$$

ρ is the density of the material, f^i are body forces, f^i_J are inertia forces, f^i_C are Coriolis forces and f^i_Z denotes the centripetal forces:

$$f^i_C = {}_{(1)}\triangle^{im}\dot{u}_m\;,\; {}_{(1)}\triangle^{im} = 2\epsilon^{inm}\,\omega_n\;,$$

$$f^i_Z = {}_{(2)}\triangle^i_m u^m\;,\; {}_{(2)}\triangle^i_m = \epsilon^{inl}\epsilon_{mnk}\,\omega_l\omega^k\;, f^i_{\mathcal{I}} = \frac{\partial^2 u^i}{\partial t^2}\;.$$

Weak form of the equations of motion:
Find a function u^i s.t.

$$\int_{{}^t\Omega}(\sigma^{kl}\sigma^i_l + \sigma^{kl}u^i_{,l})(v_{i\,,k} - \dot{u}_{i\,,k})dV - \int_{{}^t\Gamma_F} F^i(v_i - \dot{u}_i)d\gamma$$
$$- \int_{{}^t\Gamma_U} F^i(v_i - \dot{u}_i)d\gamma - \int_{{}^t\Gamma_F}\{F_N(v_N - \dot{u}_N) + \mathcal{F}_T(v_T - \dot{u}_T)\}d\gamma$$
$$+ \int_{{}^t\Omega}\rho_0(f^i_B - f^i_{\mathcal{I}} - f^i_C - f^i_Z)(v_i - \dot{u}_i)dV = 0$$

for all test functions $v_i \in V \subset W^{2,2}({}^t\Omega)$.
(Duvant and Lions, 1972)
Abstract formulation:

- Variational equation of the second order
  - + parabolic regularization (depending on a parameter λ)
  - + a priori estimates (not depending on λ)

+ $\lambda \to 0 \quad \Rightarrow$ unique solution σ^{ij}

- $\Rightarrow$ First-order Cauchy-problem in a Hilbert space V in the feedback form

$$\begin{array}{ll} \dot{z} = Az + b\varphi\,(c^* z, t), & A\text{-generator of} \\ z(0) = z_0 & \text{a semigroup} \end{array}$$

- Linear part: $\dot{z} = Az + bu(t), \quad \sigma = c^* z$

$$\chi(p) = c^*(A - pI)^{-1} b \ \text{ transfer}$$

tion

$$\chi(p) = \begin{cases} \text{rational function} & \hat{=}\, ODE \\ \text{meromorphic function} & \hat{=}\, PDE \end{cases}$$

2 Plastic buckling and flutter bifurcations in quasi-static problems

2.1 General buckling theory

Consider in $\mathbb{R}^3$ the system

$$(\mathrm{A}) \begin{cases} \varepsilon_{ij} = K_{ij}[u_k] & \text{in} \quad {}^t\Omega \quad , \\ L_\alpha\,[\sigma^{ij}\,,\,u_i\,,\,f^i] = 0 & \text{in} \quad {}^t\Omega\ ,\ \alpha = 1,2,\dots\ , \\ \tilde{L}_\beta\,[\sigma^{ij}\,,\,u_i\,,\,U_i\,,F^i] = 0 & \text{on} \quad {}^t\Gamma\ ,\ \beta = 1,2,\dots\ , \end{cases}$$

where $\{\varepsilon_{ij}, \sigma^{ij}, u_k\}$ are generalized strains, stresses and displacements.

Loads are dead loads, forces not depending on displacements.
Fix a time t_0 and consider the variational equation

$$(\mathrm{B}) \begin{cases} \delta\varepsilon_{ij} = K^0_{ij}\,[\delta u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega \\ L^0_\alpha\,[\delta\sigma^{ij}\,,\,\delta u_i\,,\,\delta f^i] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ \tilde{L}^0_\beta\,[\delta\sigma^{ij}\,,\,\delta u_i\,,\,\delta U_i\,,\delta f^i] = 0 & \text{on} \quad {}^{t_0+\delta t}\,\Gamma\ , \\ (\alpha = 1,2,\dots\ , \quad \beta = 1,2,\dots\ .) & \end{cases}$$

Plastic wrinkling is associated with non-uniqueness of solution prolongation at $t = t_0$ (Hill, 1958; Hutchinson, 1974).
Suppose $\{\bar{\varepsilon}_{ij}, \bar{\sigma}^{ij}, \bar{u}_k\}$ and $\{\bar{\bar{\varepsilon}}_{ij}, \bar{\bar{\sigma}}^{ij}, \bar{\bar{u}}_k\}$ are two solutions starting at t_0:

$$\begin{aligned} \triangle\,\varepsilon_{ij} &= \bar{\bar{\varepsilon}}_{ij} - \bar{\varepsilon}_{ij} = \delta\varepsilon_{ij}\ , \\ \triangle\,\sigma^{ij} &= \bar{\bar{\sigma}}^{ij} - \bar{\sigma}^{ij} = \delta\sigma^{ij}\ , \\ \triangle u_i &= \bar{\bar{u}}_i - \bar{u}_i \end{aligned}$$

Homogenous perturbational system

$$(\mathrm{C}) \begin{cases} \triangle\,\varepsilon_{ij} = K^0_{ij}\,[\triangle u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ L^0_\alpha\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ \tilde{L}^\alpha_\beta\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0\,,0] = 0 & \text{on} \quad {}^{t_0+\delta t}\Gamma\ , \\ \triangle\sigma^{ij} = \begin{cases} L^{ijmn}_e \cdot \triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_e \text{ (elastic)}, \\ L^{ijmn}_p \cdot \triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_p \text{ (plastic)}. \end{cases} & \end{cases}$$

(Klyushnikov, 1980)

2.2 Plastic wrinkling of shells

Averaged stresses (over the shell thickness)

$$N^{ij} = \int_{-h/2}^{h/2} \sigma^{ij} \mathrm{d}z\,,$$

Bending moments

$$M^{ij} = \int_{-h/2}^{h/2} \sigma^{ij} z \mathrm{d}z \quad (i,j \widehat{=} x,y)$$

Shearing forces

$$Q^{i} = \int_{-h/2}^{h/2} \sigma^{3i} \mathrm{d}z\ .$$

Kirchhoff-Love assumption

$$e_{ij} = \varepsilon_{ij} + z\kappa_{ij}\,, \quad z \in \left[-\tfrac{h}{2}, \tfrac{h}{2}\right]\,, \quad i,j = 1,2$$

$\kappa_{ij} = w_{,ij}\ (= \frac{\partial^2 w}{\partial x_i \partial x_j})$
Force equilibrium equations

$$\frac{\partial N^{11}}{\partial x} + \frac{\partial N^{12}}{\partial y} = 0, \quad \frac{\partial N^{22}}{\partial y} + \frac{\partial N^{21}}{\partial x} = 0\ ,$$

$$\frac{\partial Q^1}{\partial x} + \frac{\partial Q^2}{\partial y} + N^{11}\frac{\partial^2 w}{\partial x^2} + \left(N^{12} + N^{21}\right)\frac{\partial^2 w}{\partial x \partial y} + N^{22}\frac{\partial^2 w}{\partial y^2} = F^3$$

Moment equilibrium equations

$$\begin{aligned} \frac{\partial M^{11}}{\partial x} + \frac{\partial M^{12}}{\partial y} &= Q^1\,, \\ \frac{\partial M^{22}}{\partial y} + \frac{\partial M^{21}}{\partial x} &= Q^2\,. \end{aligned}$$

$$N^{ij}_{,j} = 0, \qquad M^{ij}_{,ij} + N^{ij} w_{,ij} = F^3\,.$$

Denote by $s^{ij} = \operatorname{dev}\sigma^{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ the deviator of the stress tensor, and by $\sigma_{\text{int}} = \sqrt{\frac{1}{2}\, s^{ij} s_{ij}}$ the stress intensity. G is the elastic shear modulus, E is Young's modulus, $E_s = \dfrac{\sigma_{\text{int}}}{\varepsilon_{\text{int}}}$ is the secant modulus, $G' = \dfrac{G}{1 + 3G(\frac{1}{E_s} - \frac{1}{E})}$ is the instantaneous shear modulus and α is the loading index, i.e.,
$\alpha = \begin{cases} \neq 0 & \text{in} \quad \text{plastic parts} \\ 0 & \text{in} \quad \text{elastic parts.} \end{cases}$
Material law

$$\delta\sigma^{ij} = 2\,G\,\left[\delta e^{ij} + \delta^{ij}\delta e^{kk} - \tfrac{\alpha}{2\sigma^2_{\text{int}}}\sigma^{ij}\sigma^{mn}\delta e_{mn}\right] \qquad (i,j,k = 1,2)$$

$$
\begin{aligned}
\triangle N^{ij} &= 2G\left[\int_{-h/2}^{h/2}(\triangle e^{ij}+\delta^{ij}\triangle e^{kk})\mathrm{d}z - \int\limits_{\text{plastic part}}\right. \\
\triangle M^{ij} &= 2G\left[\int_{-h/2}^{h/2}(\triangle e^{ij}+\delta^{ij}\triangle e^{kk})z\mathrm{d}z - \int\limits_{\text{plastic part}}\right. \\
\triangle e_{ij} &= \triangle\,\varepsilon_{ij}+z\triangle\kappa_{ij}\,, \\
\triangle N^{ij} &= A^{ijmn}\triangle\,\varepsilon_{mn} \quad \text{and} \quad \triangle M^{ij}=D^{ijmn}\triangle\kappa_{mn},
\end{aligned}
$$

with

$$
\begin{aligned}
A^{ijmn} &= 2Gh\left[\delta^{im}+\delta^{jn}+\delta^{ij}\delta^{mn}-\frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right], \\
D^{ijmn} &= \frac{Gh^3}{6}\left[\delta^{im}\delta^{jn}+\delta^{ij}\delta^{mn}-\frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right].
\end{aligned}
$$

Special case $N^{ij}=h\sigma^{ij}$ (constant stress over the plate)
Bifurcation equation for plastic-elastic buckling

$$
\triangle w_{\,,ijij}-\frac{1}{4}\left(1-\frac{G'}{G}\right)\frac{1}{\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle w_{\,,mnij}+\frac{3\varepsilon^{ij}}{Gh^2}\triangle w_{\,,ij} = 0\,.
$$

(Klyushnikov, 1980; Korovlev, 1971)

2.3 The plastic buckling behaviour of thin plates under constant pressure

a) **Simply supported rectangular plate**

Conditions

$$N^{22}=N^{12}=0.$$

Writing $N^{11}=-\,\tau_{\text{int}}h$ we get the bifurcation equation

$$D_2\triangle^2 w - D_3\frac{\partial^4 w}{\partial x^4}+\tau_{\text{int}}h\frac{\partial^2 w}{\partial x^2}=0$$

where w denotes the displacements in transversal to the plate direction.
Case 1: All edges are freely supported.

$$w(x,y)=A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\tau_{\text{int}}=\frac{\pi^2}{h}\left[(D_2-D_3)\frac{m^2}{a^2}+2D_2\frac{n^2}{b^2}+D_2\frac{n^4a^2}{m^2b^4}\right].$$

($n=1$ and an elastic-plastic material with $\beta=\frac{E}{E_s}$) (Korovlev, 1971)

$$e_{\text{int}}=\frac{\pi^2}{a}\frac{h^2}{b^2}\left[\frac{1+3\beta}{4}\left(\frac{b}{a}\right)^2 m^2+\left(\frac{a}{b}\right)^2\frac{1}{m^2}+2\right]\ (m\in\mathbb{N})\,.$$

Case 2: The edges $x=0$ and $x=a$ are supported and the edges $y=\pm b/2$ are free. Under these assumptions the buckling mode can be considered as

$$w(x)=A\sin\frac{m\pi x}{a}$$

Critical load for wrinkling

$$\tau_{\mathrm{int}} = \frac{\pi^2}{36}(1+3\beta)\frac{h^2}{a^2}\,.$$

b) **Circular plate under constant inplane pressure**

$$\begin{aligned}&N^{11} = N^{22} = \tau_{\mathrm{int}} h \quad \text{and} \quad N^{12} = 0 \quad \text{as}\\ &(D_2 - D_3)\triangle^2 w + \tau_{\mathrm{int}} h \triangle w = 0\,.\end{aligned}$$

$\Phi = \triangle w$

$$(D_2 - D_3)\triangle\Phi + \tau_{\mathrm{int}}\, h\Phi = 0$$

Case 1: Axisymmetric plastic buckling

$\Phi = CJ_0(r)$ where $J_0(r)$ is the Bessel function of degree 0.

Case 2: Non-axisymmetric plastic buckling

Buckling mode

$$\Phi(r,\varphi) = R(r)\cos\, n\,\varphi$$

Bifurcation equation

$$R^{''} + \frac{R'}{r} + \left(k^2 - \frac{n^2}{r^2}\right) R = 0$$

The solutions are the Bessel functions of the n-th order $R(r) = C\,J_n(r) \quad (C = \text{const})\,.$

$$\tau_{\mathrm{int}} \geq \frac{a^2k^2}{36}\,(1+3\alpha)\left(\frac{h}{a}\right)^2\,.$$

c) **Annular plate under inplane pressure**

Suppose there is given an annular plate with $\bar{\sigma}$ the effective stress in the flange, h the thickness of the flange, E the plastic buckling modulus, w the actual flange width, K a material constant. If

$$\frac{\bar{\sigma}}{E} \leq K\frac{h^2}{w^2}$$

than no flange wrinkling occurs.
(Kobayashi, 1963)

2.4 Plastic buckling and plastic flutter

Homogenous perturbational system in the presence of inertia forces (Bolotin, 1963)

$$\begin{aligned}\boldsymbol{L}\boldsymbol{u} - \frac{\partial^2\boldsymbol{u}}{\partial t^2} &= 0 \quad \text{in} \quad \Omega\times(0,T)\,,\\ \boldsymbol{M}\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_F\times(0,T)\,,\\ \boldsymbol{N}\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\times(0,T)\,,\end{aligned}$$

Associated static boundary value problem

$$\begin{aligned}\boldsymbol{L}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega\,,\\ \boldsymbol{M}\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_F\,,\\ \boldsymbol{N}\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,.\end{aligned}$$

Vibrational solutions in the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x})e^{iwt}$$

where $\boldsymbol{U}$ is an unknown function of the phase variables and $\omega \in \mathbb{C}$ is an unknown frequency. Boundary value problems

$$\begin{aligned} L\boldsymbol{U} + \omega^2\boldsymbol{U} &= 0 \quad \text{in} \quad \Omega\,, \\ M\boldsymbol{U} &= 0 \quad \text{on} \quad \Gamma_F\,, \\ N\boldsymbol{U} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,. \end{aligned}$$

Condition for the self-adjointness is

$$\int_\Omega \left\{ \left[L\boldsymbol{U}^{(1)} + \omega^2\boldsymbol{U}^{(1)}\right]\boldsymbol{U}^{(2)} - \left[L\boldsymbol{U}^{(2)} + \omega^2\boldsymbol{U}^{(2)}\right]\boldsymbol{U}^{(1)} \right\} = 0\,,$$

(For all perturbations $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}$ satisfying the boundary conditions.)
Suppose now that for a certain value ω the system has a nontrivial solution. If the system is selfadjoint the associated number ω^2 is real. In this case the loss of stability is statically: we have a buckling bifurcation in the variational system.
Consider the PDE problem

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} = 0\,.$$

Including inertia forces we come to the plastic wave equation (ρ denotes the material density)

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0\,.$$

Assume the initial conditions

$$w(0,t) = w(a,t) = 0\;,\; \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(a,t) = 0\;,\; t > 0.$$

We try to find a wave solution in the form

$$w(x,t) = W(x)e^{i\omega t}$$

where $W(x)$ is an unknown function and $\omega \in \mathbb{C}$ is a parameter to be defined.
We receive the ODE problem

$$(D_2 - D_3)W^{(4)} + \tau_{\text{int}} h W^{(2)} + \rho(-\omega^2)W = 0$$

or

$$W^{(4)} + k^2 W^{(2)} - \widehat{\omega}^2 W = 0\,.$$

With the abbreviations

$$k = \sqrt{\frac{\tau_{\text{int}} h}{D_2 - D_3}} \text{ and } \widehat{\omega} = \omega\sqrt{\frac{\rho}{D_2 - D_3}}$$

the critical intensity τ^*_{int} for dynamic plastic wrinkling is

$$\tau^*_{\text{int}} = {k^*}^2\,\frac{(D_2 - D_3)}{h}\,.$$

2.5 3.5 ODE model for the impact-contact problem

(Kirdeev et al., 1984)
$m\ddot{y}_1 + c(y_1 + \Delta) + \kappa(y_1 - y_2) = 0,$
$m_1\ddot{y}_2 + c_1 y_2 + \kappa(1,5y_2 - y_1 - 0,5y_3 - 0,5y_4) = m_1 e\omega^2 \sin(\omega t + \varphi),$
$m\ddot{y}_3 + c(y_3 - \Delta) + \kappa(y_3 - 0,5y_2) = 0,$
$m\ddot{y}_4 + c(y_4 + \Delta) + \kappa(y_4 - 0,5y_2) = 0.$

3 Dynamic buckling

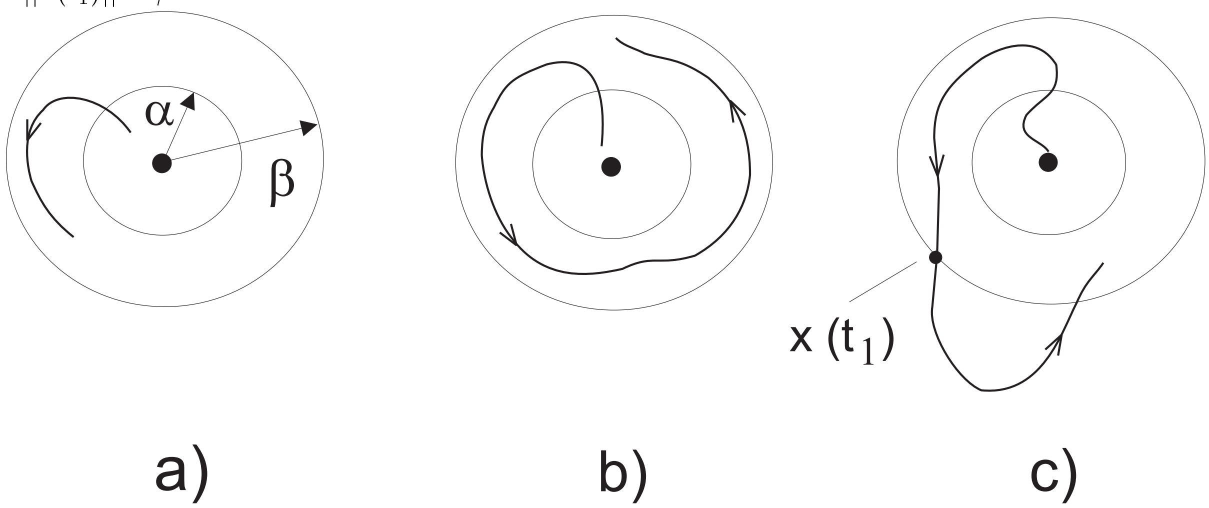
Given
$$\dot{x} = f(t, x) \tag{4.1}$$

in the Banach space B with $|| \cdot ||,\ t \in \mathcal{J} = [t_0, t_0 + T), \quad T < +\infty$

a) Def. (stability on a finite time interval)
(4.1) is stable w. r. t. $(\alpha, \beta, t_0, T, || \cdot ||), \alpha \leq \beta$, if for every solution $x(t)$ of (4.1), $||x(t_0)|| < \alpha$ implies $||x(t)|| < \beta$ for all $t \in \mathcal{J}$.

b) Lyapunov stable: $\forall \varepsilon > 0 \,\exists\, \delta > 0 :$
$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0$

c) (4.1) is unstable w. r. t. $(\alpha, \beta, t_0, T, || \cdot ||)$,
$\alpha \leq \beta$, if there exists a solution $x(t)$ of (4.1) with $||x(t_0)|| < \alpha$, and a value of time $t_1 \in (t_0, t_0 + T)$ s.t. $||x(t_1)|| = \beta$.



Lyapunov stable $\not\Rightarrow$ stability on a finite time interval;
Stability on a finite time interval $\not\Rightarrow$ Lyapunov stability
Chetaev, 1935; Kamenkov, 1953; La Salle, Lefschetz, 1961,
T. Kapitaniak, J. Brindley: Practical stability of chaotic attractors. Chaos, Solitons and Fractals 9 (1998), 43 - 50.

Theorem (Weiss & Infante, 1965) (4.1) is stable with respect to $(\alpha, \beta, t_0, T, || \cdot ||), \beta > \alpha$, if there exist $V(t, x) \in C^0 \times C^1$ and an integrable function φ on $\mathcal{J}$ s. t.

$$(i)\ \dot{V}(t, x) < \varphi(t) \quad \forall x \in \bar{B}(\beta) \backslash B(\alpha), \forall \alpha \in \mathcal{J};$$

$$(ii) \int_{t_1}^{t_2} \varphi(t) dt \leq \min_{||x||=\beta} V(t_2, x) - \max_{||x||=\alpha} V(t_1, x)$$

$$\forall t_1 < t_2, t_1, t_2 \in \mathcal{J}.$$

Remark No requirement of definiteness on such functions or their derivative.
Given $\ddot{q}^r = P^r(t, q^1, \ldots, q^n; \dot{q}^1, \ldots, \dot{q}^n)$
Variational equation

$\ddot{\xi}^r = \triangle P^r(\xi^1, \dots, \xi^n; \dot{\xi}^1, \dots, \dot{\xi}^n)$
Stability on a finite time interval $[0, t_{cr})$:

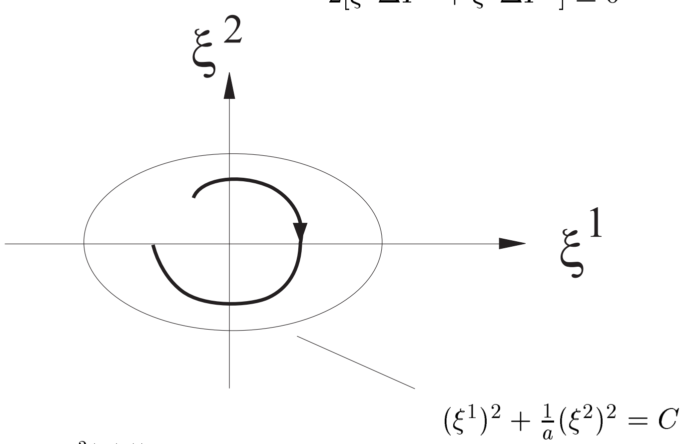
$$\delta_{ij} \triangle P^i \xi^j \equiv \sum_{i=1}^{n} \triangle P^i \xi^i < 0 \quad \text{for } 0 < t < t_{cr}$$

Loss of stability = bifurcation

$$\delta_{ij} \triangle P^i \tilde{\xi}^i > 0 \quad \text{for } t > t_{cr} \text{ and at least}$$

for one perturbed solution $\tilde{\xi}^i$
Dynamic bifurcation functional
$Q(u^i, \ddot{u}^j, t, \lambda) = \int\limits_{^t\Omega} \delta_{mn} \triangle P^m u^n dV = \int\limits_{^t\Omega} \rho \delta_{mn} \ddot{u}^m u^n dV =$
$= \int\limits_{^t\Omega} \left[\frac{1}{2} {}^0\sigma^{kl} u^m_{,k} u_{m,l} + \frac{1}{2}\sigma^{kl}\varepsilon_{kl}\right] dV +$
$\int\limits_{^t\Omega} [{}^0\sigma^{kl} u^m_{,l} + \sigma^{kl}(\delta^m_l + {}^0u^m_{,l})]_{,k} dv - \int\limits_{^t\Gamma_F} \frac{1}{2} F^m u_m d\gamma$
$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k} + {}^0u^m_{,k} u_{m,l} + {}^0u^m_{,l} u_{m,k})$
Example (variational system)
A dynamical system $\{\varphi^t\}_{t \in I}$ is stable on $[t_0, T)$ with respect to a regular $n \times n$ matrix function $S(x)$ if for all sufficiently small $\rho > 0$ from
$(S(x_0)x_0, S(x_0)x_0) \leq \rho^2$ it follows that
$(S(\varphi^t(x_0))\varphi^t(x_0), S(\varphi^t(x_0))\varphi^t(x_0)) \leq \rho^2$ for all $t \in [t_0, T)$.
Special case: $\varphi^1 = f, S(x) = p(x)I$,
$0 < p_1 \leq p(x) \leq p_2 \quad \forall x \in U \subset \mathbb{R}^n$
Harmonic oscillator
$\ddot{\xi} + a\,\xi = 0 \quad , a > 0$
$\xi^1 = \xi, \quad \xi^2 = \dot{\xi}$
$\dot{\xi}^1 = \xi^2 = \Delta P^1$
$\dot{\xi}^2 = -a\,\xi^1 = \Delta P^2$
$V(\xi^1, \xi^2) = (\xi^1)^2 + \frac{1}{a}(\xi^2)^2$
$\dot{V} = 2\xi^1(\xi^2) + \frac{1}{a}\xi^2(-a\,\xi^1) =$

$$2[\xi^1 \Delta P^1 + \xi^2 \Delta P^2] \equiv 0$$



Stability condition: $\frac{p^2(f(x))}{p^2(x)} ||Df^* Df|| < 1$
$||Df(x)^* Df(x)|| = \alpha_1^2(x)$,
$\alpha_1(x) \geq \dots \geq \alpha_n(x)$ singular values of $Df(x)$
$\Rightarrow$ Stability condition $\frac{p(f(x))}{p(x)} \alpha_1(x) < 1$ for all $x \in U$

$\omega_d(x) := \alpha_1(x)\alpha_2(x)\cdots\alpha_{d_0}(x)\alpha^s_{d_0+1}(x)$
singular value function
$d = d_0 + s,\ d_0 \in \{0, 1, \ldots, n-1\}, s \in [0, 1)$
Generalisation $\frac{p(f(x))}{p(x)}\omega_d(x) < 1$

$$\forall x \in K \subset U,$$

where $K \subset U$ is a compact f-invariant set
$\Rightarrow \dim_F K < d$
$\Rightarrow h_{\text{top}}(f) \leq \max_{x\in K}\{0, \ln ||Df(x)||\} \dim_F K$
ODE case
$\dot{x} = f(x)\ ,\ V : U \to \mathbb{R}\ ,\ U \supset K\ , \varphi^\tau(K) = K$
$\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ are the ordered eigenvalues of $\frac{1}{2}[Df(x)^* + Df(x)]$

$$\exists T > 0 : \int_0^T [\dot{V}(\varphi^\tau(x)) + \lambda_1(\varphi^\tau(x)) + \cdots + s\lambda_{d_0}(\varphi^\tau(x)) + \lambda_{d_0+1}(\varphi^\tau(x))]d\tau < 0$$

$\Rightarrow \dim_F K < d.$

Conclusions

1. Determination of critical parameters $\{t_{cr}, \lambda_{cr}\}$ and displacements by means of the bifurcation functional Q

2. Postbuckling behaviour / Calculation of the bifurcated trajectory (displacement field)

- Use of the variational principle from continuum mechanics

Durant / Lions, 1972
Lee, L.H. N. / Ni, C. M., 1973

$$\boxed{\inf_{u^m\in V}\left\{\int_{t_{cr}\Omega}\left[\sigma^{kl}\ddot{\varepsilon}_{kl} + \frac{1}{2}\rho\,\ddot{u}_m\,\ddot{u}^m - \rho f^m\ddot{u}_m\right]dV - \int_{t_{cr}\Gamma_F} F^k\ddot{u}_k\,d\gamma\right\}}$$

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k} + u^m_{,k}\cdot u_{m,l}),$$

$$\dot{\sigma}^{kl} = \frac{\partial W}{\partial\dot{\varepsilon}_{kl}}\ ,\ W = \text{strain - rate potential}$$

$$2W = \begin{cases} L^{klmn}\dot{\varepsilon}_{kl}\dot{\varepsilon}_{mn}\ , & b^{mn}\dot{\varepsilon}_{mn} \leq 0 \\ L^{klmn}\dot{\varepsilon}_{kl}\dot{\varepsilon}_{mn}\ -\ (b^{mn}\dot{\varepsilon}_{mn})^2\ , & b^{mn}\dot{\varepsilon}_{mn} \leq 0 \end{cases}$$

- Galerkin method / wavelet approximation $\Rightarrow$ finite dimensional optimization problem

3. Time series analysis for determining the (nonlinear) bifurcation type (saddle node, Hopf, static wrinkling or elastic-plastic wave etc.)

Example: $\dot{x} = f(x, \lambda)$, $f(0,0) = 0$,
eigenvalues of $\frac{\partial f}{\partial x}(0,0)$:
$\varepsilon_{1,2}(\lambda)_{|\lambda=0} = \pm i\omega$, $\omega \in \mathbb{R}$, $\omega \neq 0$,

$$\mathrm{Re}\,\varepsilon_k \neq 0 \ , \ k = 3, \ldots , n$$

normal form: $\dot{r} = \lambda r - L r^3$,
$(n = 2)$ $\dot{\varphi} = g(\varphi, r)$

$L \neq 0 \Rightarrow$ Hopf bifurcation

If the right-hand part f is available $\Rightarrow$ computation of L
If not: $L > 0 \Leftrightarrow \frac{d\mathrm{Re}\varepsilon_{1,2}(\lambda)}{d\lambda}|_{\lambda=0} > 0 \curvearrowright$ can be measured by varying λ

Existence results for plastic and frictional contact problems
Duvant & Lions, 1972; Moreau, 1976; Johnson, 1976; Ciort & Rabier, 1980; Nećas & Hlaváćek, 1981; Temam, 1985; Rabier et al., 1986; Monteiro Marcques, 1994;

Basic results for elasto-plastic stability and wrinkling of shells
Hill, 1958; Il'yushin, 1963; Korovlev, 1971; Hutchinson, 1974; Klyushnikov, 1980; Palmov, 1998;

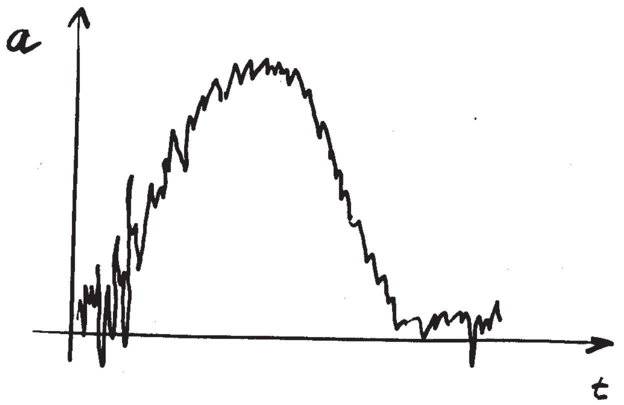
Non-linear shell theory
Mushtari, 1957; Vlasov, 1958; Donnell, 1976;

Elasto-plastic analysis of flange wrinkling in deep drawing process
Energy methods
Geckeler, 1928; Yu & Johnson, 1982; Yossifon & Tirosh, 1984; Yang & Lee, 1992; Cao, 1999
Hill's bifurcation theory
Fatnassi et al., 1984; Naruse, 1986; Améziane-Hassani & Neale, 1990; Wang et al., 1994; Scherzinger & Triantafyllidis, 2000; Chu & Xu, 2001;

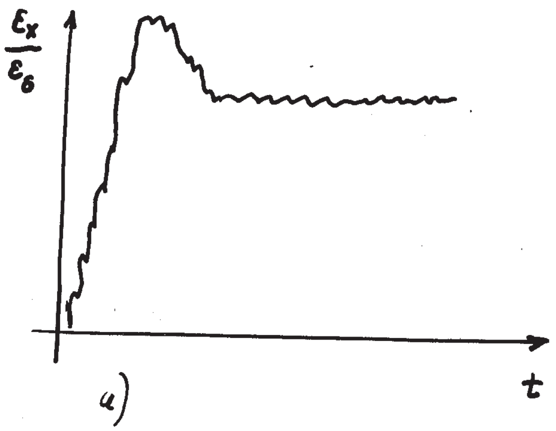
Convential sheet metal spinning
Instability and wrinkling
(bifurcation analysis and energy methods)
Siebel & Dröge, 1954; Reichel, 1958; Avitzur & Yang, 1960; Kolpakciaglu, 1961; Kegg, 1961; Kobayashi, 1963; Wells, 1968; Barkaya, 1974; Kirdeev et al., 1984; Korol'kov, 2001;
Roller pass programming
Mogil'nyi, 1972; Hayama et al., 1991; Korol'kov et al., 1999;
Statistic and time-series analysis
Mogil'nyi & Moisseev, 1979; Kiryanov & Mishunin, 1997; Suliman et al., 2000; Malenichev & Val'ter, 2001;

Stability of a spinning disc with a transverse concentrated load
(Coriolis effect, gyroscopic problems, divergence instability, resonance, dynamic buckling, circumferential waves)
Carlin et al., 1975; Iwan & Moeller, 1976; Padovan, 1978; Sprinivasan & Ramamurti, 1980; Nowinski, 1983; Leung & Pinnington, 1987; Chen & Wong, 1994; Chen & Jhu, 1997; Huang & Kuang, 2001;

Dynamic behavior of oscillatiors with clearance and periodically time-varying forces
(structures with gaps and impacting, chaotic resonance)
Peterka, 1974; Panovka, 1977; Choi & Noah, 1992; Lenci et al., 1994; Goldman & Muszyusha, 1994; Kahraman & Blankenship, 1997;

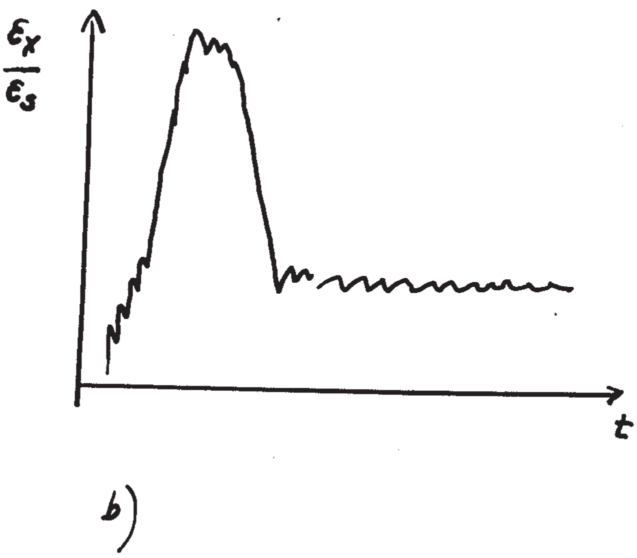


Measurement of acceleration on a plate



Measurement of strain histories on a plate

a) front surface



b) back surface

L. Zhu: Stress and strain analysis of plates subjected to transverse wedge impact J. of Strain Analysis, **31**, 1, 1 - 6, 1996.