Plastic wrinkling and flutter in sheet metal spinning

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Abstract

In this report a short mathematical and mechanical description of the sheet metal spinning process is given. The representation is based on nonlinear shell theory, elasto-plastic material laws and perturbation theories. Various plastic buckling and plastic flutter situations are analyzed. The influence of the angular velocity of a spinning disc on the plasticity process is investigated.

1 Introduction

In this report we discuss some mathematical and mechanical basic problems arising in the investigation of instabilities in metal spinning processes. Plastic and dynamical wrinkle formations in metal forming processes for quasi-static deformations such as deep-drawing are intensively investigated in the last time ([2, 7, 8, 29, 33, 39, 55, 57, 58, 59, 60]). The central theoretical approach is based on bifurcation functionals which go back to Hill and Hutchinson ([24, 25]). In most of these investigations it is assumed that the forces are conservative. Inertia forces and general gyroscopic forces are not considered. A next step in the characterization of bifurcation effects in solid mechanics was done by Klyushnikov ([34]). The central part in this theory is the investigation of uniqueness of homogenous variational problems. In a number of papers the energy method is used to guarantee that the considered regular deformation remains stable ([39]). The idea is that for the formation of plastic wave structures a minimal membrane energy is necessary. For the spinning process the energy method is used by Kobayashi ([35]) in order to derive sufficient conditions for suppressing of wrinkling.

The plastic wrinkling problem for spinning processes is in general a non-conservative dynamical problem. The bifurcation scenario is richer than in the static case. In the present report we describe the qualitative nature of the spinning process as mechanical contact problem. We discuss some basic bifurcation mechanisms which occur in the general and complex bifurcation scenario. The main consequence of this discussion is that the loss of stability in a spinning disc is possible either as static (plastic) buckling (divergence instability) or as dynamic (plastic) buckling (flutter instability). In order to see which type of stability loss occurs, a free boundary value problem has to be solved. It remains to show (using center manifold theory or Lyapunov-Schmidt reduction) that the linear bifurcation results are applicable for the full nonlinear buckling problem. Important for this is the property that the multiplicity of the critical eigenvalue of the linearized problem for annular plates or curved shells is finite.

The report is organized as follows. In Ch.2 we give a mathematical and mechanical description of the spinning process. Basic equilibrium equations are derived from the large deformation theory in continuum mechanics (Sect.2.1). In Sect.2.2 elasto-plastic material laws are introduced. Using the von Mises-plasticity approach in Sect.2.3 we investigate the influence of the angular velocity on the plastification of a spinning disc. In the same section the minimal stress intensity necessary to plastify an annular disc is estimated. For the full mechanical model of the spinning process the friction between roller and rotating blank has to be described. A general friction theory based on the plasticity approach is introduced in Sect.2.4. Using this friction theory, the large deformation theory from Sect.2.1 and the elasto-plastic material laws from Sect.2.2 we derive in Sect.2.5 the principal contact problem of sheet metal spinning as a free boundary-value problem. The last two sections of Ch.2 are devoted to the linearization of this contact problem and a short characterization of the influence of temperature on the deformation process.

The central bifurcation results are presented in Ch.3. In Sect.3.1 we give a short introduction in plastic bifurcation theory which goes back to Hill ([24]) and Klyushnikov ([34]). This general approach is applied to plastic plate wrinkling in Sect.3.2. In Sect.3.3 we analyze some model buckling situations which are important for the spinning process. The estimation of critical stresses for loaded plates and the representation of concentrated loads are considered in Sect.3.4 and Sect.3.5. In Sect.3.6 we characterize the different ways of loss of stability in the spinning process. Since the system is in general non-conservative, generically the loss of stability is associated with plastic buckling (divergence instability) or (and) plastic flutter (plastic Hopf bifurcation). This characterization depends on the multiplicity of the critical eigenvalue of the bifurcation equation. The most dangerous situation for the spinning process is the combined form "divergence-flutter" of stability loss. Although this is a non-generic bifurcation it shows the theoretical sources for chaotic in time and fractal in space behaviour ("turbulence" in elasto-plastic material).

In the last chapter we continue the mathematical description of the spinning process as large non-symmetric deformation of nonlinear shells under concentrated loads. This model contains all possibilities for further bifurcation analysis of the spinning process.

2 The spinning process as contact problem

2.1 Basic facts from finite-deformation theory

The wrinkling formation in metal spinning is a global deformation process depending on time. For the geometrical description of this dynamical process are strongly nonlinear deformation ([14, 40]) and shell theories ([47, 56, 15]) necessary. For a short description we use the representation from [14].

Suppose that the deformation is a one-parametric family of maps $\{\Phi^t\}_{t\in[0,T]}$ defined on an open set $\Omega \subset \mathbb{R}^3$ as

$$\boldsymbol{\Phi}^t : \Omega \to \mathbb{R}^3 \,. \tag{2.1}$$

In local coordinates the map (2.1) can be written as

$$x^i = x^i\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t\right) \quad , \quad i = 1, 2, 3 \quad , \quad t \in [0, T]\,. \tag{2.2}$$

Let us assume that an orthogonal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is introduced in $\mathbb{R}^3$. The deformation tensor $\boldsymbol{F}^t$ in the point $\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3\right)$ with respect to this basis is defined by

$$\boldsymbol{F}^t = \frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\, \boldsymbol{e}_i \boldsymbol{e}_j \,. \tag{2.3}$$

(Here and in the following Einstein summation convention is used.)The adjoint to this tensor is

$$(\boldsymbol{F}^t)^* = \frac{\partial x^i}{\partial \overset{\circ}{x}{}^j} \boldsymbol{e}_j \boldsymbol{e}_i \,.$$

Important metric information about the deformation process contains the tensor

$$\overset{\circ}{\Lambda}{}^t = \sqrt{(\boldsymbol{F}^t)^* \cdot \boldsymbol{F}^t}\,. \tag{2.4}$$

The eigenvalues λ_i^t of $\overset{\circ}{\Lambda}{}^t$ are the singular values of the deformation tensor. Thus, if B_ε is a small ball of radius ε in $\mathbb{R}^3$ then the image $\boldsymbol{F}^t(B)$ is an ellipsoid with half-lengths $\lambda_i^t\varepsilon$ $(i = 1, 2, 3)$ and the volume $\mathrm{vol}_3(\boldsymbol{F}^t(B)) = J^t \mathrm{vol}_3(B)$, where $J^t = \lambda_1^t\lambda_2^t\lambda_3^t$.

Suppose that $S \subset \Omega$ is a small surface with outer normal $\boldsymbol{n}(\boldsymbol{x})$ in the point $\boldsymbol{x} \in S$ and S' is the surface after deformation under $\boldsymbol{F}^t$. Then the force acting on a small part dS' from the positive normal side is defined by $\sigma^i_j(\boldsymbol{x}) n_i dS$, where n_i are the components of $\boldsymbol{n}(\boldsymbol{x})$ and σ^i_j is the first Piola-Kirchhoff tensor. If the columns of $\left(\frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\right)$ are linearly independent we can write $\sigma^i_j = \sigma^{il} \frac{\partial x^l}{\partial \overset{\circ}{x}{}^j}$. The so defined tensor σ^{il} is the second Piola-Kirchhoff tensor (we identify the components of the tensor in the fixed basis with the tensor).

Assume now that the map (2.2) has the representation

$$x^i = \overset{\circ}{x}{}^i + u^i(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t) \tag{2.5}$$

where u^i is a smooth displacement function. Introduce the small strain tensor e_{ij} by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \overset{\circ}{x}{}^j} + \frac{\partial u_j}{\partial \overset{\circ}{x}{}^i}\right) \tag{2.6}$$

and the Lagrange strain tensor ε_{ij} by

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2} \frac{\partial u_k}{\partial \overset{\circ}{x}{}^i} \frac{\partial u_k}{\partial \overset{\circ}{x}{}^j} \,. \tag{2.7}$$

If the deformations are small we can assume $\overset{\circ}{x}{}^i = x^i$ and $\overset{\circ}{\nabla} = \nabla$ and the equilibrium equation with body forces f_j and material density ρ can be written as

$$\frac{\partial \sigma_{\alpha j}}{\partial x^\alpha} + \rho \left(f_j - \frac{\partial^2 u_j}{\partial t^2} \right) = 0 \,. \tag{2.8}$$

Let us suppose now that our system is described in curvilinear coordinates, i.e. the cartesian coordinates of any point can be written as $x^i = x^i(\xi^1, \xi^2, \xi^3)$, where $(\xi^1, \xi^2, \xi^3) \in G$. Suppose that $\boldsymbol{R}_i = \frac{\partial x^j}{\partial \xi^i} \boldsymbol{g}_i$ where $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ are the basis vectors with respect to the coordinate lines. Then $g_{ij} = \boldsymbol{R}_i \cdot \boldsymbol{R}_j$ defines the metric tensor, $\epsilon_{ijk} = \boldsymbol{R}_i \cdot (\boldsymbol{R}_j \times \boldsymbol{R}_k)$ resp. $\epsilon^{ijk} = \boldsymbol{R}^i \cdot (\boldsymbol{R}^1 \times \boldsymbol{R}^k)$ are the Levi-Cività tensors and

$$\Gamma^k_{ij} = \frac{1}{2} \left(\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l} \right) g^{lk} \tag{2.9}$$

are the Christoffel symbols.

The second (symmetric) Kirchhoff-Piola tensor in curvilinear coordinates can be written as $J\sigma^{\alpha\beta} \overset{\circ}{\boldsymbol{R}}_\alpha \overset{\circ}{\boldsymbol{R}}_\beta$. ($\circ$ denotes the computation in the undeformed part of the body.)

Now we can define the contravariant differentiation procedure through

$$\begin{aligned} u_{j,i} \equiv \nabla_i u_j &= \frac{\partial u_j}{\partial \xi^i} - \Gamma^k_{ij} u_k \quad \text{and} \\ u^j_{\ ,i} \equiv \nabla_i u^j &= \frac{\partial u^j}{\partial \xi^i} + \Gamma^j_{ki} u^k \,. \end{aligned} \tag{2.10}$$

The small strain and finite strain tensors are given through

$$e_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) \tag{2.11}$$

and

$$\varepsilon_{ij} = \frac{1}{2} (g_{ij} - \overset{\circ}{g}_{ij}) = e_{ij} - \frac{1}{2} g^{\alpha\beta} \nabla_i u_\alpha \nabla_j u_\beta \,. \tag{2.12}$$

2.2 Constitutive law

It is assumed that the material of the blank is elastic-plastic. The plasticity domain K on the blank is defined as $\{\sigma^{ij} : \mathcal{H}(\sigma^{ij}) \le 0\}$ where $\mathcal{H}$ is a given scalar function (yield function). Note the following two important examples:

a) *von Mises material:* $\mathcal{H}(\sigma^{ij}) = \frac{1}{2} s^{ij} s_{ij} - k^2$

with $s^{ij} = s_{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ as deviator of σ^{ij} and $k \neq 0$ is a constant.

b) *Tresca material:* $\mathcal{H}(\sigma^{ij}) = \max|\sigma_i - \sigma_j| - k$

where the maximum is computed over all eigenvalues of the tensor σ^{ij} and $k > 0$ is again a constant.

We assume that the total strain is at any point of the body the sum of an elastic part and a plastic part, i.e.

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}. \tag{2.13}$$

The elastic part is computed by the generalized Hooke's law

$$\varepsilon^e_{ij} = L^e_{ijkl}\,\sigma^{kl} \tag{2.14}$$

where L^e_{ijkl} is the elastic deformation tensor. For the plastic part we have $\varepsilon^p_{ij} \neq 0$ if and only if $\mathcal{H}(\sigma^{ij}) \leq 0$. In the deformation theory the computation rule for ε^p_{ij} is

$$\varepsilon^p_{ij} = \frac{1}{2\mu}\frac{\sqrt{\frac{1}{2}\,s_{ij}s^{ij}} - k}{\sqrt{\frac{1}{2}\,s_{ij}s^{ij}}} \tag{2.15}$$

where s_{ij} is again the deviator stress and μ is a material constant. In the flow theory (which includes the history of the material) we set

$$\dot{\varepsilon}^p_{ij} = \lambda\frac{\partial Y}{\partial\sigma^{ij}} \tag{2.16}$$

where λ is a factor of proportionality and Y is a given potential. If the flow rule is used we can write

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}. \tag{2.17}$$

For further details about material laws see [25, 26, 34, 40, 43, 51, 19].

2.3 Plasticity zone in the spinning disc and in the annular disc

Let us consider at first a circular disc of thickness h and density ρ rotating with constant angular velocity ω. The stability and deformation properties of a spinning disc are well understood ([10, 11, 12, 13, 16, 41]). Most of the investigations are concerned with elastic materials. In this subsection we demonstrate how the plasticity properties of a spinning disc depend on the angular velocity. The rotation is assumed around the z-axis. Suppose that (r, φ, z) are the cylindrical coordinates and τ_r and τ_φ are the non-zero Cauchy stress components which are defined (inertia effects are not considered) on the basis of (2.8) by the quasi-static differential equation

$$\frac{\partial\tau_r}{\partial r} + \frac{\tau_r - \tau_\varphi}{r} = -\rho\,\omega^2 r\,. \tag{2.18}$$

In order to describe the development of plasticity in the disc we have to solve a free boundary problem which is connected with the unknown boundary of the plasticity domain around the centre of rotation. We want to use the von Mises-plasticity law together with (2.13) possessing the yield condition

$$\tau_r^2 - \tau_r \tau_\varphi + \tau_\varphi^2 = k^2 \qquad (k = \text{const})\,. \tag{2.19}$$

The plasticity region can be written as ring $0 \le r \le r_*$. We want to applicate the deformation theory in order to describe the connection between Cauchy stresses and small strains e_r and e_φ, i. e.,

$$\begin{aligned} e_r &= \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\,(\tau_r - \tau)\,, \\ e_\varphi &= \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\,(\tau_\varphi - \tau)\,. \end{aligned} \tag{2.20}$$

In the last formula $\tau = \frac{1}{3}(\tau_r + \tau_\varphi)$ is the averaged stress, $\Psi = \Psi(r)$ is an unknown function, E is Young's modulus and ν is Poisson's ratio.

Suppose that the radius of the disc is r_0 and introduce the new variables

$$\rho = \frac{r}{r_0} \quad , \quad \tau_\rho = \frac{\tau_r}{k} \quad \text{and} \quad \tau_\varphi := \frac{\tau_\varphi}{k} \tag{2.21}$$

and the constants

$$a = \frac{r_*}{r_0} \quad , \quad \alpha = \frac{E}{k} \quad \text{and} \quad \lambda = \frac{\rho\,\omega^2 r_0^2}{k}\,. \tag{2.22}$$

It follows that equation (2.18) transforms to the equilibrium equation

$$\frac{\partial \tau_\rho}{\partial \rho} + \frac{\tau_\rho - \tau_\varphi}{\rho} = -\lambda\rho \quad , \quad 0 \le \rho \le 1\,, \tag{2.23}$$

and (2.19) to the yield condition

$$\tau_\rho^2 - \tau_\rho \tau\varphi + \tau_\varphi^2 = 1\,. \tag{2.24}$$

Taking the ansatz (see [3])

$$\begin{aligned} \tau_\rho &= \frac{2}{\sqrt{3}} \cos\left(\Phi + \frac{\pi}{6}\right)\,, \\ \tau_\varphi &= \frac{2}{\sqrt{3}} \cos\left(\Phi + \frac{\pi}{6}\right) \end{aligned} \tag{2.25}$$

with the unknown function $\Phi = \Phi(\rho)$ we see that (2.24) is automatically satisfied and (2.23) transforms to the ODE problem

$$\rho\,\frac{d\,\Phi}{d\rho} = \frac{\lambda\rho^2 \frac{\sqrt{3}}{2} - \sin\Phi}{\sin(\Phi + \frac{\pi}{6})} \tag{2.26}$$

with initial condition $\tau_\rho = \tau_\varphi$ for $\rho = 0$. In addition to (2.26) we have to satisfy on the whole disc the compatibility condition

$$\frac{\partial\, e_\rho}{\partial \rho} + \frac{e_\varphi - e_\rho}{\rho} = 0\,, \tag{2.27}$$

which guarantees the continuity of the solutions. Equation (2.27) is equivalent to the linear ODE problem

$$\frac{1+\nu}{2\alpha\sqrt{3}}\,(\sqrt{3}\sin\omega + \cos\omega)\,\frac{d\Psi}{d\rho} + \Psi\left[\frac{1+\nu}{\alpha\rho}\sin\omega + \frac{\sqrt{3}}{2}\frac{d\Psi}{d\rho}\,\left(\sqrt{3}\cos\omega - \sin\omega\right)\right] =$$
$$= \frac{1-2\nu}{\sqrt{3}\alpha}\sin\omega\frac{d\Psi}{d\rho}\,.$$

For the determination of stresses in the elastic domain $a \le \rho \le 1$ we use Hooke's law (2.14) in the form

$$\begin{aligned} e_\rho &= \frac{1}{2}\,(\tau_\rho - \nu\tau_\varphi)\,,\\ e_\varphi &= \frac{1}{2}\,(\tau_\varphi - \nu\tau_\rho)\,. \end{aligned} \tag{2.28}$$

A necessary condition for the solution results in

$$\frac{\partial\tau_\rho}{\partial\rho} - \nu\frac{\partial\tau_\rho}{\partial\rho} + (1+\nu)\,\frac{\tau_\varphi - \tau_\rho}{\rho} = 0 \tag{2.29}$$

together with the initial condition $\tau_{\rho_{|\rho=1}} = 0$. This gives, with an arbitrary constant $C \in \mathbb{R}$, the general solution

$$\begin{aligned} \tau_\rho &= C\Big(1 - \frac{1}{\rho^2}\Big) + \frac{\lambda(3+\nu)}{8}\,(1-\rho^2)\,,\\ \tau_\rho &= C\Big(1 + \frac{1}{\rho^2}\Big) + \frac{\lambda}{8}\,[\,3+\nu-(1+3\,\nu)\rho^2]\,. \end{aligned} \tag{2.30}$$

For $\rho = a(\rho)$ we have to guarantee the continuity of the radial and tangential stresses. The last property implies the free boundary condition

$$\begin{aligned} C\Big(1 - \frac{1}{a^2}\Big) + \frac{\lambda(3+\nu)}{8}(1-a^2) &= \frac{2}{\sqrt{3}}\cos\left(\Phi(a) + \frac{\pi}{6}\right),\\ C\Big(1 + \frac{1}{a^2}\Big) + \frac{\lambda}{8}\,[\,3+\nu-(1+3\nu)a^2] &= \frac{2}{\sqrt{3}}\cos\left(\Phi(a) - \frac{\pi}{6}\right). \end{aligned} \tag{2.31}$$

From the nonlinear system (2.31) follows the existence of a solution
$C = C(\lambda,\nu)$, $a = a(\lambda,\nu)$ ([3]). Recall that λ is a linear function of ω^2 given by (2.22).

Using the property $\rho = \dfrac{u_\rho}{\varepsilon_\varphi}$ we can write our final result in the form

$$
\begin{aligned}
\tau_\rho &= C\Big(1 - \frac{e_\varphi^2}{u_\rho^2}\Big) + \frac{\lambda(3+\nu)}{8}\Big(1 - \frac{u_\rho^2}{e_\varphi^2}\Big)\,, \\
\tau_\rho &= C\Big(1 + \frac{e_\varphi^2}{u_\rho^2}\Big) + \frac{\lambda}{8}\left[3 + \nu - (1+3\nu)\frac{u_\rho^2}{e_\varphi^2}\right].
\end{aligned}
\tag{2.32}
$$

Let us consider an annular plate of thickness h, of outer radius b and of inner radius a. Suppose that this plate is clamped at the inner edge with the outer edge free subjected to uniform radial compression p at the inner edges (Fig. 2.1)

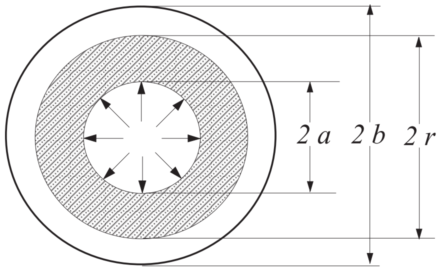

Figure 2.1: The annular plate and the plastic and elastic domains

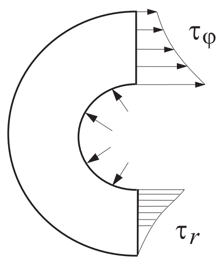

Figure 2.2: Stresses in the annular plate

For the exactness of the deformation theory it is in the following assumed ([36]) that

$$\frac{a}{b} \geq 0.37 \tag{2.33}$$

On the basis of the Lamé formula we can compute the tangential and radial stresses depending on the actual radius r by ([19])

$$\tau_r = \frac{p\,a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{2.34}$$

and

$$\tau_\varphi = \frac{p\,a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)\,. \tag{2.35}$$

(Fig. 2.2).

The stress intensity is ([36])

$$\tau_{\text{int}} = \frac{3\beta}{4}\,(\tau_\varphi - \tau_r) \tag{2.36}$$

with $\beta = \frac{2+\sqrt{3}}{2}$.

According to the von Mises-plasticity law it is possible to compute an inner plastic ring domain $a \le r \le r_o$ and an outer elastic ring domain $r_o \le r \le b$. The value r_o characterizes the free boundary. Let us shortly consider two cases.

a) Elastic zone:
$$\tau_r^e = -\frac{2E}{3}A\left(\frac{1}{r^2} - \frac{1}{b^2}\right) , \tag{2.37}$$
$$\tau_\varphi^e = -\frac{2E}{3}A\left(\frac{1}{r^2} + \frac{1}{b^2}\right) , \tag{2.38}$$

where τ_r^e and τ_φ^e with an arbitrary constant A are solutions of the equilibrium equation.

b) Plastic zone:
$$\tau_r^p = \frac{4k}{3\beta}\left(\ln\frac{r}{a} - \kappa\right) , \tag{2.39}$$
$$\tau_\varphi^p = \frac{4k}{3\beta}\left(\ln\frac{r}{a} - \kappa + 1\right) \tag{2.40}$$

with $\kappa = \frac{3p\beta}{4k}$. The continuity and compatibility conditions lead to

$$\tau^e_{r_{|r=r_o}} = \tau^p_{r_{|r=r_o}} \tag{2.41}$$

and

$$\tau_\varphi^e - \tau_r^e = \frac{4k}{3\beta} . \tag{2.42}$$

It is easy to see that the fully developed plastic annular ring is defined with a critical pressure

$$p_{\text{cr}} = \frac{4k}{3p}\ln\frac{a}{b} . \tag{2.43}$$

Recall that $a_s := \frac{b}{a}$ is the spinning ratio in metal forming process.

2.4 Friction theory

Frictional properties have to be considered when the tangential part of the motion is important in the response of two bodies coming into contact during the dynamic spinning process. For a short introduction into friction models we use the representation from ([46]) (see also [9, 17, 20, 28, 40]).

For two bodies Ω_1 and Ω_2 in contact define on one body Ω_1 the outward normal vector $\boldsymbol{n}$ at the point of contact. We separate the relative displacement $\boldsymbol{u}$ between the two bodies into the tangential and normal parts

$$\boldsymbol{u} = \boldsymbol{u}_T + u_N\boldsymbol{n} , \tag{2.44}$$

where $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_T = (\mathrm{id} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{u}$. Similarly, for the surface stress $\boldsymbol{p}$ we have

$$\boldsymbol{p} = \boldsymbol{p}_T + p_N \boldsymbol{n} \ , \tag{2.45}$$

where p_N is just the contact pressure.

The tangential relative velocity $\dot{\boldsymbol{u}}_T$ is decomposed into the adherence part and the slipping part

$$\dot{\boldsymbol{u}}_T = \dot{\boldsymbol{u}}_T^{ad} + \dot{\boldsymbol{u}}_T^{sl} \ . \tag{2.46}$$

Since the adherence stage is elastic, for simplicity we assume a linear elasticity law

$$\boldsymbol{p}_T = -k \boldsymbol{u}_T^{ad} \tag{2.47}$$

where k is the elastic contact stiffness.

For the slipping part we define, as in plasticity, a simple evolution rule

$$\dot{\boldsymbol{u}}_T^{sl} = -\dot{\gamma} \frac{\partial \Psi}{\partial \boldsymbol{p}_T} \quad , \tag{2.48}$$

where the slip potential Ψ determines the direction of slip, and γ is a real function. The state of friction is determined by the slip function Φ and the loading and unloading conditions

$$\begin{array}{lll} \text{loading:} & \Phi > 0 \quad , & \dot{\gamma} = 0 \, , \\ \text{unloading:} & \Phi = 0 \quad , & \dot{\gamma} > 0 \end{array}$$

and

$$\begin{array}{ll} \Phi \leq 0 \Rightarrow & \text{adherence,} \\ \Phi > 0 \Rightarrow & \text{slipping.} \end{array}$$

For a general model of friction using this approach we define the frictional stress at the contact point $\mathcal{F}(p_N) = \mu |p_N|$, where μ is the coefficient of friction. Then the slip function is

$$\Phi(\boldsymbol{p}_T, p_N) = \|\boldsymbol{p}_T\| - \mu |p_N| \tag{2.49}$$

and, for isotropic friction, the direction of friction apposes slip, giving the slip potential as

$$\Psi(\boldsymbol{p}_T) = \|\boldsymbol{p}_T\|.$$

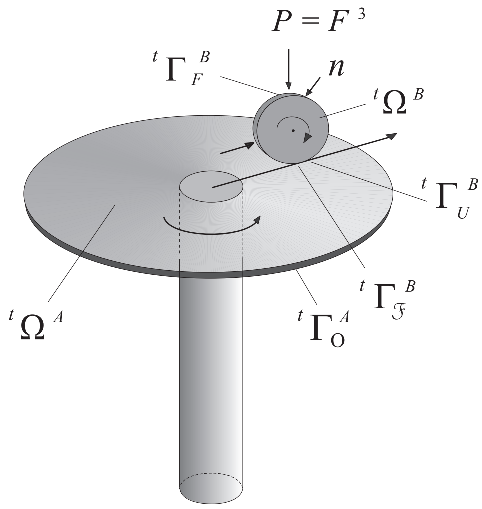

Figure 2.3: Blank and roller at time t

2.5 Large deformation dynamic elastic-plastic contact problem

Let us consider the mechanical system "blank-roller" (Fig. 2.3) as a contact problem of two bodies A and B. Suppose that for any considered time t the whole body ${}^t\Omega = {}^t\Omega^A \cup {}^t\Omega^B$ is the union of the two bodies and ${}^t\Gamma = \partial\, {}^t\Omega = \partial\, {}^t\Omega^A \cup \partial\, {}^t\Omega^B = {}^t\Gamma^A \cup {}^t\Gamma^B$ is their boundary.

For the large deformation elastic-plastic contact problems in $\mathbb{R}^3$ the fundamental systems equation with typical boundary and initial conditions are the following ones ([46, 42]):

(i) Equilibrium equations

$$(\sigma^{kl}\delta_l^i + \sigma^{kl}u^i_{,l})_{,k} + \rho f^i = \rho f^i_{\mathcal{I}} + \rho f^i_C + \rho f^i_Z, \qquad t \in (0,T), \quad \boldsymbol{x} \in {}^t\Omega\,. \tag{2.50}$$

(ii) Kinematic boundary conditions (prescribed displacements)

$$u_i(\boldsymbol{x},t) = U_i(\boldsymbol{x},t), \qquad \boldsymbol{x} \in {}^t\Gamma_U = {}^t\Gamma_U^A \cup {}^t\Gamma_U^B, \qquad t \in (0,T)\,. \tag{2.51}$$

(iii) Prescribed boundary forces

$$[\sigma^{kl}\delta_l^i + \sigma^{kl}u^i_{,l}]n_k = F^i, \qquad \boldsymbol{x} \in {}^t\Gamma_F^B, \qquad t \in (0,T)\,. \tag{2.52}$$

(n_1, n_2, n_3 components of $\boldsymbol{n}$)

(iv) Tangential frictional stress

$$\sigma^{ij}n_j - \sigma^{jk}n_j n_k n^i = \mathcal{F}^i, \qquad \boldsymbol{x} \in {}^t\Gamma_{\mathcal{F}}, \qquad t \in (0,T)\,. \tag{2.53}$$

(v) Initial conditions

$$u_i(\boldsymbol{x},0) = U_{0i}(\boldsymbol{x}), \qquad \dot{u}_i(x,0) = U_{1i}(\boldsymbol{x})\,. \tag{2.54}$$

In (2.50) ρ is the density of the material, f^i are body forces, f^i_J are inertia forces, f^i_C are Coriolis forces and f^i_Z denotes the centripetal forces:

$$f^i_C = {}_{(1)}\triangle^{im}\dot{u}_m \ , \ {}_{(1)}\triangle^{im} = 2\epsilon^{inm}\,\omega_n \ , \tag{2.55}$$

$$f^i_Z = {}_{(2)}\triangle^i_m u^m \ , \ {}_{(2)}\triangle^i_m = \epsilon^{inl}\epsilon_{mnk}\,\omega_l\omega^k \ , f^i_J = \frac{\partial^2 u^i}{\partial t^2} \ . \tag{2.56}$$

Let us write for completeness the weak form of the equations of motion (2.50)which is needed in FE methods. This is done by taking the scalar product of the equations of motion and $v - \dot{u}$, where v is an arbitrary test function:

$$\int\limits_{^t\Omega} (\sigma^{kl}\sigma^i_l + \sigma^{kl}u^i_{,l})(v^i_{,k} - \dot{u}^i_{,k})dV - \int\limits_{^t\Gamma_F} F^i(v^i - \dot{u}^i)d\gamma$$
$$- \int\limits_{^t\Gamma_U} F^i(v^i - \dot{u}^i)d\gamma - \int\limits_{^t\Gamma_F} \{F_N(v_N - \dot{u}_N) + \mathcal{F}_T(v_T - \dot{u}_T)\}d\gamma$$
$$+ \int\limits_{^t\Omega} \rho_0(f^i_B - f^i_{\mathcal{I}} - f^i_C - f^i_Z)dV = 0. \tag{2.57}$$

The weak formulation (2.57) of the contact problem (2.50)-(2.54) is widely used in FE methods ([6, 9, 20, 30, 31]).

2.6 Pertubational equations of motion

Let us introduce the following perturbations in the displacement and stress fields. For this we assume that the functions in the original equation (2.50) are indicated with $\sim$:

$$\tilde{u}_m(\boldsymbol{x}, t) = {}^{(o)}u_m(\boldsymbol{x}) + u_m(\boldsymbol{x}, t) \ , \tag{2.58}$$

$$\tilde{\sigma}^{mn} = {}^{(o)}\sigma^{mn}(\boldsymbol{x}) + \tau^{mn}(\boldsymbol{x}, t) + {}^{(o)}\tau^{mn}(\boldsymbol{x}, t) \ , \tag{2.59}$$

$$\tilde{\varepsilon}_{mn} = {}^{(o)}\varepsilon_{mn}(\boldsymbol{x}) + e_{mn}(\boldsymbol{x}, t) + {}^{(o)}e_{mn}(\boldsymbol{x}, t) \ , \tag{2.60}$$

$$\varepsilon_{mn} = e_{mn}(\boldsymbol{x}, t) + {}^{(o)}e_{mn}(\boldsymbol{x}, t) \ , \tag{2.61}$$

$${}^{(o)}\sigma^{mn}(\boldsymbol{x}) = L^{mnkl}_{EP}\ {}^{(o)}\varepsilon_{kl}(\boldsymbol{x}) \ , \tag{2.62}$$

$$\tau^{mn} = L^{mnkl}e_{kl} \ , \quad {}^{(o)}\tau^{mn} = L^{mnkl}\ {}^{(o)}e_{kl} \ , \tag{2.63}$$

$$ {}^{(o)}\varepsilon_{mn} = \frac{1}{2}\left[{}^{(o)}u_{m,n} + {}^{(o)}u_{n,m}\right] + \frac{1}{2}\,{}^{(o)}u^{l}_{\ ,m}\,{}^{(o)}u_{l,m} \ , \tag{2.64} $$

$$ e_{mn} = \frac{1}{2}\left[u_{m,n} + u_{n,m}\right] , \quad {}^{(o)}e_{mn} = \frac{1}{2}\left[{}^{(o)}u^{l}_{\ ,m}u_{l,n} + {}^{(o)}u^{l}_{\ ,n}u_{l,m}\right] \ . \tag{2.65} $$

Since the initial fields are static, the prescribed tractions, displacements and body forces take the form

$$ \tilde{f}^{p}_{B}(\boldsymbol{x},t) = {}^{(o)}f^{p}_{B}(\boldsymbol{x}) + f^{p}_{B}(\boldsymbol{x},t) \ , \tag{2.66} $$
$$ \tilde{F}^{\,i}(\boldsymbol{x},t) = {}^{(o)}F^{i}(\boldsymbol{x},t) + F^{\,i}(\boldsymbol{x},t) \ , \tag{2.67} $$
$$ \mathcal{F}^{i}(\boldsymbol{x},t) = {}^{(o)}\mathcal{F}^{i}(\boldsymbol{x},t) + \mathcal{F}^{i}(\boldsymbol{x},t) \ . \tag{2.68} $$

In the terms of (2.58) - (2.68) equation (2.50) reduces to the following perturbational gyroscopic equations of motion (see [42])

$$ (\sigma^{mn} + {}^{(o)}\sigma^{mn}u^{p}_{\ ,n})_{,m} + f^{p}_{B} = \rho\,\ddot{u}^{p} + {}_{(1)}\triangle^{pm}\dot{u}_{m} + {}_{(2)}\triangle^{i}_{m}u^{m} \ , \tag{2.69} $$

where $\sigma^{mp} = \tau^{mp} + {}^{(o)}\tau^{mp} + \tau^{mn}\,{}^{(o)}u^{p}_{\ ,n} + {}^{(o)}\tau^{mn(o)}u^{p}_{\ ,n}$.

The boundary and initial conditions associated with (2.69) are given by

$$ (i) \quad u_i = U_i \qquad \text{on} \quad \Gamma_U \times (0,T) \ , \tag{2.70} $$
$$ (ii) \quad \sigma^{mp}n_m = F^{p} \qquad \text{on} \quad \Gamma_U \times (0,T) \ , \tag{2.71} $$
$$ (iii) \quad \sigma^{ij}n_j - \sigma^{jk}n_jn_kn^{i} = \mathcal{F}^{i} \qquad \text{on} \quad \Gamma_{\mathcal{F}} \times (0,T) \ , \tag{2.72} $$
$$ (iv) \quad u_i(\boldsymbol{x},0) = \boldsymbol{U}_{oi}(\boldsymbol{x}) \ , \quad \dot{u}_i\,(\boldsymbol{x},0) = \boldsymbol{U}_{1i}(\boldsymbol{x}) \qquad \text{on} \quad \Omega \ . \tag{2.73} $$

Note that the equations (2.69)-(2.73) define the canonical linearized system of a contact problem. Existence and stability properties of such systems are discussed in [17, 40, 42].

2.7 Thermal effects

Let us show how to include the influence of the temperature field $\Theta(\boldsymbol{x},t)$ into material laws. It is known ([26]) that heat energy coming in our model from the friction process represents 15 - 20 % of the whole energy.

The govering equation for the temperature is the heat equation

$$ \frac{\partial\Theta}{\partial t} = k\triangle\Theta + w(\boldsymbol{u}) \tag{2.74} $$

depending on the actual displacement $\boldsymbol{u}$ of the system (2.50) (w is a certain nonlinear function) and satisfying certain boundary and initial conditions. If we have a small-strain nonlinear elasto-plastic material law of the type

$$ \tau^{ij} = \frac{2\,\Phi(e_{\text{int}})}{3\,e_{\text{int}}}(e_{ij} - e\delta_{ij}) + 3Ke\delta_{ij} \tag{2.75} $$

where $e = \frac{1}{3}e_{ii}$, $e_{\rm int}$ is the strain intensity, K is a constant and Φ is a nonlinear material function, the influence of Θ may be written as $\Phi = \Phi(e_{\rm int}, \Theta)$ and (2.75) goes over into ([26])

$$\tau^{ij} = \frac{2\Phi(e_{\rm int}, \Theta)}{3\, e_{\rm int}}(e_{ij} - e\delta_{ij}) + 3K(e - \alpha\,\Theta)\delta_{ij} \tag{2.76}$$

where $\alpha > 0$ is a certain parameter.

Note that the introduction of the temperature field into the system (2.50) has a positive influence on the stability properties of the whole system. This is connected with the special properties of the Laplace operator in equation (2.74).

3 Plastic buckling and flutter bifurcations in quasi-static problems

3.1 General buckling theory

Consider in $\mathbb{R}^3$ the system

$$(\mathrm{A})\begin{cases} \varepsilon_{ij} = K_{ij}[u_k] & \text{in} \quad {}^t\Omega \quad , \\ L_\alpha\,[\sigma^{ij}\,,\, u_i\,,\, f^i] = 0 & \text{in} \quad {}^t\Omega\,,\ \alpha = 1, 2, \dots \quad , \\ \tilde{L}_\beta\,[\sigma^{ij}\,,\, u_i\,,\, U_i\,, F^i] = 0 & \text{on} \quad {}^t\Gamma\,,\ \beta = 1, 2, \dots \quad , \end{cases} \tag{3.1}$$

where $\{\varepsilon_{ij}, \sigma^{ij}, u_k\}$ are generalized strains, stresses and displacements, respectively. For any t we have with ${}^t\Omega$ and ${}^t\Gamma$ denoted the time-dependent area and the associated boundary. We assume that (A) is a formal representation of the contact problem (2.50)-(2.54) considered as quasi-static system.

Assumption: All loads are dead loads, i.e., forces not depending on displacements.

Fix a time t_0 and consider the variational equation to (3.1)

$$(\mathrm{B})\begin{cases} \delta\varepsilon_{ij} = K^0_{ij}\,[\delta u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega \\ L^0_\alpha\,[\delta\sigma^{ij}\,,\, \delta u_i\,,\, \delta f^i] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\,,\ \alpha = 1, 2, \dots\,, \\ \tilde{L}^0_\beta\,[\delta\sigma^{ij}\,,\, \delta u_i\,,\, \delta U_i\,, \delta f^i] = 0 & \text{on} \quad {}^{t_0+\delta t}\,\Gamma\,,\ \beta = 1, 2, \dots\,. \end{cases} \tag{3.2}$$

For the bifurcation analysis of (3.2) we assume that plastic wrinkling is associated with non-uniqueness of solution prolongation at $t = t_0$ ([24, 25]). Suppose for this that $\{\bar{\varepsilon}_{ij}, \bar{\sigma}^{ij}, \bar{u}_k\}$ and $\{\bar{\bar{\varepsilon}}_{ij}, \bar{\bar{\sigma}}^{ij}, \bar{\bar{u}}_k\}$ are two solutions of (3.2) starting at t_0 in the same point. Compute the values

$$\begin{aligned} \triangle\,\varepsilon_{ij} &= \bar{\bar{\varepsilon}}_{ij} - \bar{\varepsilon}_{ij} = \delta\varepsilon_{ij}\,, \\ \triangle\,\sigma^{ij} &= \bar{\bar{\sigma}}^{ij} - \bar{\sigma}^{ij} = \delta\sigma^{ij}\,, \\ \triangle u_i &= \bar{\bar{u}}_i - \bar{u}_i \end{aligned} \tag{3.3}$$

and consider the following homogenous perturbational system

$$
(\,\mathrm{C}\,)\left\{\begin{array}{ll}
\qquad\qquad \triangle\,\varepsilon_{ij} = K^0_{ij}\,[\triangle u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega\,, \\
\qquad\qquad L^0_\alpha\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\,, \\
\qquad\qquad \tilde{L}^\alpha_\beta\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0\,,0] = 0 & \text{on} \quad {}^{t_0+\delta t}\Gamma\,, \\
\triangle\sigma^{ij} = \left\{\begin{array}{lll} L^{ijmn}_e\cdot\triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_e & \text{(elastic part)}, \\ L^{ijmn}_p\cdot\triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_p & \text{(plastic part)}. \end{array}\right. &
\end{array}\right. \tag{3.4}
$$

It follows ([34]) that plastic wrinkling in (3.2) is possible if we show the existence of non-zero solutions $(\triangle\varepsilon_{ij}, \triangle\sigma^{ij}, \triangle u_i)$ of the problem (C) at a certain time t_0.

3.2 Plastic wrinkling of plates

In this section we demonstrate some applications of the general bifurcation theory to plate buckling problems which are connected with the spinning process. For the description of the dynamics of a plate we need the averaged stresses (over the plate thickness)

$$N^{ij} = \int_{-h/2}^{h/2} \sigma^{ij}\mathrm{d}z\,, \tag{3.5}$$

the bending moments

$$M^{ij} = \int_{-h/2}^{h/2} \sigma^{ij} z\mathrm{d}z \quad (i, j \mathrel{\widehat{=}} x, y) \tag{3.6}$$

and the shearing forces

$$Q^i = \int_{-h/2}^{h/2} \sigma^{3i}\mathrm{d}z\ . \tag{3.7}$$

For simplicity we take the assumptions $F^1 = F^2 = 0$ with respect to the components of the forcing term. Denote by u_1, u_2 the in-plane displacements and by w the out-of-plane displacements. It follows that the small stress tensor for the middle surface is $\varepsilon_{ij} = \frac{1}{2}\,[u_{i,j} + u_{j,i}]$. Assume the Kirchhoff-Love assumption for the plate, i.e.,

$$e_{ij} = \varepsilon_{ij} + z\kappa_{ij}\,, \quad z \in \left[-\tfrac{h}{2}, \tfrac{h}{2}\right]\,, \quad i, j = 1, 2 \tag{3.8}$$

where $\kappa_{ij} = w_{,ij}$ ($= \frac{\partial^2 w}{\partial x_i \partial x_j}$ in Cartesian coordinates).

Consider the force equilibrium equations

$$\frac{\partial N^{11}}{\partial x} + \frac{\partial N^{12}}{\partial y} = 0, \quad \frac{\partial N^{22}}{\partial y} + \frac{\partial N^{21}}{\partial x} = 0\ ,$$

$$\frac{\partial Q^1}{\partial x} + \frac{\partial Q^2}{\partial y} + N^{11}\frac{\partial^2 w}{\partial x^2} + \left(N^{12} + N^{21}\right)\frac{\partial^2 w}{\partial x \partial y} + N^{22}\frac{\partial^2 w}{\partial y^2} = F^3 \tag{3.9}$$

and the two moment equilibrium equations

$$\begin{aligned} \frac{\partial M^{11}}{\partial x} + \frac{\partial M^{12}}{\partial y} &= Q^1 \,, \\ \frac{\partial M^{22}}{\partial y} + \frac{\partial M^{21}}{\partial x} &= Q^2 \,. \end{aligned} \tag{3.10}$$

Excluding in (3.9), (3.10) Q^i and using the symmetries $N^{ij} = N^{ji}$ and $M^{ij} = M^{ji}$ we derive the system (see [34])

$$N^{ij}_{,j} = 0, \qquad M^{ij}_{,ij} + N^{ij} w_{,ij} = F^3 \,. \tag{3.11}$$

We get in generalized coordinates N^{ij}, M^{ij} ("stresses"), $\varepsilon_{ij}, \kappa_{ij}$ ("strains"), u_i (displacements) and material laws a generalized system (A) of Sect. 3.1. The perturbation system (B) with δF^i, $\delta F^3, \delta N^{ij}, \delta M^{ij}, \delta W, \delta\varepsilon_{ij} = \frac{1}{2}\{\delta u_{ij} + \delta u_j i\}, \delta\kappa_{ij} = \delta w_{,ij}$ $(i,j = 1,2)$ follows from (3.11) as

$$\begin{aligned} \delta N^{ij}_{,j} &= \delta F^i \,, \\ \delta M^{ij}_{,ij} + N^{ij}\delta w_{,ij} + w_{,ij}\,\delta N^{ij} &= \delta F^3 \,. \end{aligned} \tag{3.12}$$

Suppose now that we have at a certain time at least two solutions $\left(\delta\bar{M}^{ij},\, \delta\bar{N}^{ij},\, \delta\bar{u}_{ij}\right)$ and $\left(\delta\bar{\bar{M}}^{ij},\, \delta\bar{\bar{N}}^{ij},\, \delta\,\bar{\bar{u}}_{ij}\right)$ of the variational system (3.12). Thus the values

$$\triangle M^{ij} = \delta\bar{\bar{M}}^{ij} - \delta\bar{M}^{ij} \,,\ \triangle N^{ij} = \delta\bar{\bar{N}}^{ij} - \delta\bar{N}^{ij} \tag{3.13}$$

$$\triangle u_i = \delta\bar{\bar{u}}_i - \delta\bar{u}_i \quad \text{and} \quad \triangle w = \delta\bar{\bar{w}} = \delta\bar{w} \tag{3.14}$$

satisfy the following system

$$\begin{aligned} \triangle N^{ij}_{,j} &= 0 \,, \quad \triangle M^{ij}_{,ij} + N^{ij}\triangle w_{,ij} + w_{,ij}\triangle N_{ij} = 0 \,, \\ \triangle\varepsilon_{ij} &= \frac{1}{2}\left[\triangle u_{i,j} + \triangle u_{j,i}\right] , \quad \triangle\kappa_{ij} = \triangle w_{,ij} \,, \end{aligned} \tag{3.15}$$

in which for simplicity is assumed that $F^3 = 0$. Denote by $s^{ij} = \operatorname{dev}\sigma^{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ the deviator of the stress tensor, and by $\sigma_{\text{int}} = \sqrt{\frac{1}{2}\, s^{ij} s_{ij}}$ the stress intensity. In an analog manner let us define for ε_{ij} the strain intensity ε_{int}. Recall that G is the elastic shear modulus, E is Young's modulus, $E_s = \dfrac{\sigma_{\text{int}}}{\varepsilon_{\text{int}}}$ is the secant modulus, $G' = \dfrac{G}{1 + 3G(\frac{1}{E_s} - \frac{1}{E})}$ is the instantaneous shear modulus and α is the loading index, i.e., $\alpha = \begin{cases} \neq 0 & \text{in} \quad \text{plastic parts} \\ 0 & \text{in} \quad \text{elastic parts.} \end{cases}$

We assume that the material law has the form

$$\delta\sigma^{ij} = 2\,G\,\left[\delta e^{ij} + \delta^{ij}\delta e^{kk} - \tfrac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\delta e_{mn}\right] \quad (i,j,k=1,2) \tag{3.16}$$

(e^{ij} are the contravariant components of e_{ij}). It follows that

$$\begin{aligned}
\triangle N^{ij} &= 2\,G\left[\int_{-h/2}^{h/2}(\triangle e^{ij} + \delta^{ij}\triangle e^{kk})\mathrm{d}z - \int\limits_{\text{plastic part}} \frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle e^{mn}\mathrm{d}z\right],\\
\triangle M^{ij} &= 2\,G\left[\int_{-h/2}^{h/2}(\triangle e^{ij} + \delta^{ij}\triangle e^{kk})z\mathrm{d}z - \int\limits_{\text{plastic part}} \frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle e^{mn}z\mathrm{d}z\right],\\
\triangle e_{ij} &= \triangle\,\varepsilon_{ij} + z\triangle\kappa_{ij}\,,\\
\triangle N^{ij} &= A^{ijmn}\triangle\,\varepsilon_{mn} \quad \text{and} \quad \triangle M^{ij} = D^{ijmn}\triangle\kappa_{mn},
\end{aligned}$$

with

$$\begin{aligned}
A^{ijmn} &= 2\,Gh\left[\delta^{im} + \delta^{jn} + \delta^{ij}\delta^{mn} - \frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right], \qquad (3.17)\\
D^{ijmn} &= \frac{Gh^3}{6}\left[\delta^{im}\delta^{jn} + \delta^{ij}\delta^{mn} - \frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right].
\end{aligned}$$

If we use the assumptions $F^3=0$, $Q^i=0$, $w_{\,,ij}=0$ (the state before perturbations) a non-trivial solution of the variational system is possible only for the system

$$\triangle M^{ij}_{\,,ij} + N^{ij}\triangle w_{\,,ij} = 0\,. \tag{3.18}$$

This implies the equations

$$\triangle M^{ij}_{\,,ij} = (D^{ijmn}\underbrace{\triangle\kappa_{mn}}_{=\triangle w_{\,,mn}})_{\,,ij} = \tag{3.19}$$

$$= \left(\frac{Gh^3}{6}\left[\delta^{im}\delta^{jn} + \delta^{ij}\delta^{mn} - \frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right]\triangle w_{\,,mn}\right)_{,ij} \tag{3.20}$$

and

$$\frac{Gh^3}{6}\left[2\triangle w_{\,,ijij} - \left(\frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle w_{\,,mn}\right)_{,ij}\right] + N^{ij}\triangle w_{\,,ij} = 0\,. \tag{3.21}$$

If we consider the special case $N^{ij} = h\sigma^{ij}$ (constant stress over the plate) we get the bifurcation equation ([3, 34, 36]) for plastic-elastic buckling

$$\triangle w_{\,,ijij} - \frac{1}{4}\left(1 - \frac{G'}{G}\right)\frac{1}{\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle w_{\,,mnij} + \frac{3\varepsilon^{ij}}{Gh^2}\triangle w_{\,,ij} = 0\,. \tag{3.22}$$

In the special case $\alpha = 0$ we derive from this the elastic buckling equation

$$\triangle w_{,ijij} + \frac{3\sigma^{ij}}{Gh^2}\triangle w_{,ij} = 0\,. \tag{3.23}$$

Equations (3.22) and (3.23) are fundamental for the elasto-plastic buckling theory of plates. The same type of bifurcation equations can be derived for nonlinear shells.

3.3 The plastic buckling behaviour of thin plates under constant pressure

We want to applicate the bifurcation results of Sect.3.2 to the following model situations which are important in sheet metal spinning process. For simplicity we use small strains and stresses.

a) **Simply supported rectangular plate**

Consider at first a rectangular plate of thickness h under constant inplane pressure $\boldsymbol{p}$ as it is shown in Fig. 3.1

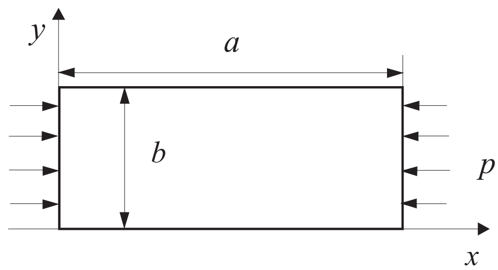

Figure 3.1: A rectangular plate under constant inplane pressure

It follows that in the present situation the averaged stresses are

$$N^{22} = N^{12} = 0.$$

Writing $N^{11} = -\,\tau_{\text{int}}h$ we get from (3.23) the bifurcation equation

$$D_2\triangle^2 w - D_3\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}}h\frac{\partial^2 w}{\partial x^2} = 0 \tag{3.24}$$

where w denotes the displacements in transversal to the plate direction. Note that in the purely elastic case equation (3.24) reduces to the well-known elastic buckling equation

$$D_2\triangle^2 w + \tau_{\text{int}}h\frac{\partial^2 w}{\partial x^2} = 0\,. \tag{3.25}$$

In the following analysis of the buckling behaviour of the plate we distinguish two cases ([36]).

Case 1: All edges are freely supported. For the linear bifurcation analysis we take an ansatz of the buckling modes as

$$w(x,y) = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{3.26}$$

with A an arbitrary real constant and m, n natural numbers. Inserting (3.26) into equation (3.25) we receive the eigenvalues

$$\tau_{\text{int}} = \frac{\pi^2}{h} \left[(D_2 - D_3)\frac{m^2}{a^2} + 2D_2 \frac{n^2}{b^2} + D_2 \frac{n^4 a^2}{m^2 b^4} \right] . \tag{3.27}$$

(In this notation the dependence on m and n is not marked.)

In the elastic buckling situation and with $n = 1$ we derive from (3.27) the well-known formula

$$\tau_{\text{int}} = \frac{\pi^2 D_2}{h\, b^2} \left[\frac{b\, m}{a} + \frac{a}{m\, b} \right]^2 .$$

For $n = 1$ and an elastic-plastic material with $\beta = \frac{E}{E_s}$ we can calculate the critical load for the plate as

$$e_{\text{int}} = \frac{\pi^2}{a} \frac{h^2}{b^2} \left[\frac{1+3\beta}{4} \left(\frac{b}{a}\right)^2 m^2 + \left(\frac{a}{b}\right)^2 \frac{1}{m^2} + 2 \right] \quad (m \in \mathbb{N}) . \tag{3.28}$$

The plastic buckling modes for $m = 1$ and $m = 2$ are shown in Fig. 3.2 .

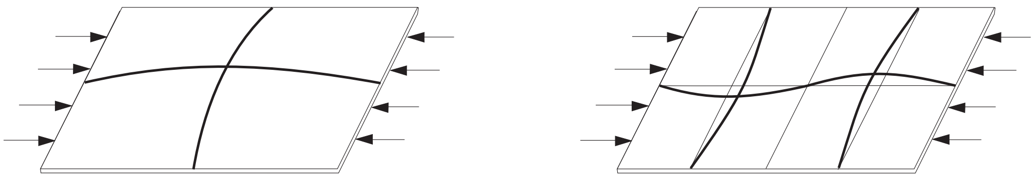

Figure 3.2: The first buckling modes

Case 2: The edges $x = 0$ and $x = a$ are supported and the edges $y = \pm b/2$ are free. Under these assumptions the buckling mode can be considered as

$$w(x) = A \sin \frac{m\pi x}{a} \tag{3.29}$$

with a constant A.

The critical load for wrinkling is now (see [36])

$$\tau_{\text{int}} = \frac{\pi^2}{36}(1+3\beta)\frac{h^2}{a^2} . \tag{3.30}$$

If only one of the edges $y = \pm b/2$ is free, the critical load for plastic wrinkling is larger. It follows that if in the last case

$$\tau_{\text{int}} < \frac{\pi^2}{36}(1+3\beta)\frac{h^2}{a^2} \tag{3.31}$$

plastic wrinkling is impossible.

b) **Circular plate under constant inplane pressure**

Consider the circular plate which is given in Fig. 3.3. The bifurcation equation

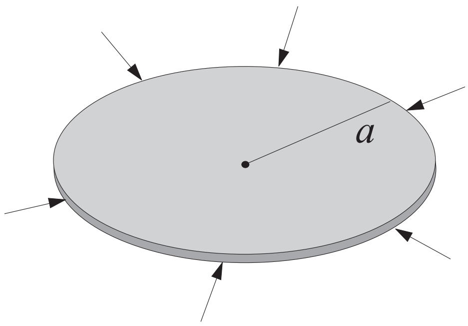

Figure 3.3: Circular plate under constant pressure

is derived from (3.23) with

$$N^{11} = N^{22} = \tau_{\text{int}} h \quad \text{and} \quad N^{12} = 0 \quad \text{as}$$
$$(D_2 - D_3)\triangle^2 w + \tau_{\text{int}} h \triangle w = 0\,. \tag{3.32}$$

Introducing a new function Φ through $\Phi = \triangle w$ we see that if Φ is a solution of the equation

$$(D_2 - D_3)\triangle\Phi + \tau_{\text{int}}\, h\Phi = 0 \tag{3.33}$$

then Φ is also a solution of (3.32).

In the further analysis we distinguish two cases.

Case 1: Axisymmetric plastic buckling

The solution Φ of (3.33) can be written as $\Phi = CJ_0(r)$ where $J_0(r)$ is the Bessel function of degree 0. A principal buckling behaviour of the circular plate in this case shows Fig. 3.4.

Case 2: Non-axisymmetric plastic buckling

Let us write the buckling mode as

$$\Phi(r,\varphi) = R(r)\cos n\varphi \tag{3.34}$$

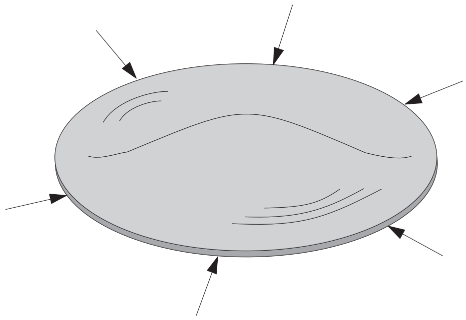

Figure 3.4: Axisymmetric buckling of a circular plate

with the unknown function R and the natural number n. The bifurcation equation (3.23) with respect to R becomes

$$R'' + \frac{R'}{r} + \left(k^2 - \frac{n^2}{r^2}\right) R = 0 \tag{3.35}$$

where $k^2 = \dfrac{36\, e_{\text{int}}}{(1+3\,\alpha)\, h^2}$. The solutions of (3.35) are the Bessel functions of the n-th order $R(r) = C\, J_n(r) \quad (C = \text{const})$.

It follows that if ka is the smallest rooth of the characteristic equation the critical load for non-axisymmetric buckling satisfies the inequality

$$\tau_{\text{int}} \geq \frac{a^2 k^2}{36}\,(1+3\alpha)\left(\frac{h}{a}\right)^2 . \tag{3.36}$$

Note that non-axisymmetric bifurcation of a plate with and without a hole can be analyzed using the principle of virtual work ([39]).

c) **Annular plate under inplane pressure**

We restrict ourselves to the outer pressure situation shown in Fig. 3.5.

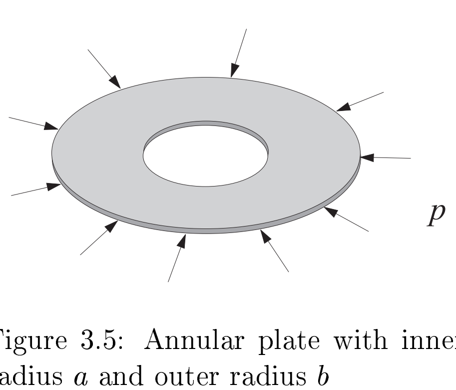

Figure 3.5: Annular plate with inner radius a and outer radius b

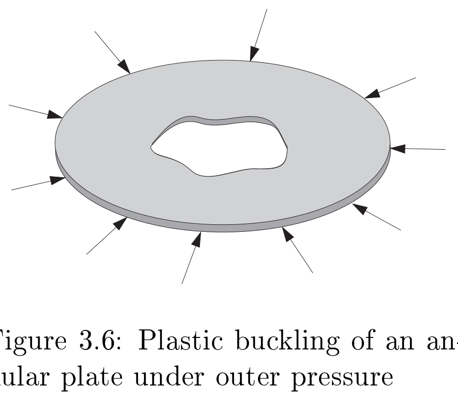

Figure 3.6: Plastic buckling of an annular plate under outer pressure

The bifurcation equation arising from (3.23) reduces (see [38]) to the PDE problem (with $b = 1$)

$$(D_2 - D_3)\,\triangle^2 w + \frac{p}{1-a^2}\left[\left(1+\frac{a^2}{r^2}\right)\triangle w - \frac{2a^2}{r^2} w_{,rr}\right] = 0\,. \tag{3.37}$$

The principal plastic buckling behaviour is shown in Fig. 3.6 .

3.4 Estimation of the critical stress intensity for loaded plates

Let us shortly describe the method which can be used for many types of deformations in order to calculate the critical wrinkling loads for loaded shells or plates. Suppose for simplicity that we have an annular plate as in Sect. 3.2 with an initial strain intensity $\overset{\circ}{e}_{\text{int}}$. Suppose also that at a certain moment the load of this plate is increased. In this situation it is necessary to consider the material law Φ in a shifted coordinate system as shown in Fig. 3.7 . It is assumed that $\tau_{\text{int}} = \Phi(e_{\text{int}})$ and

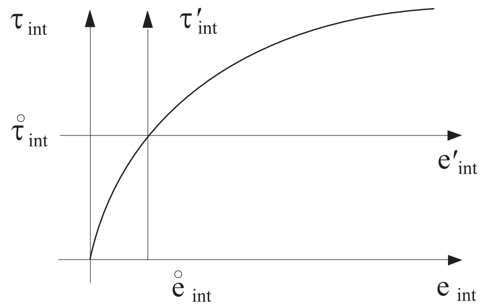

Figure 3.7: The shifted coordinate system of the material law

$\overset{\circ}{\tau}_{\text{int}} = \Phi(\overset{\circ}{e}_{\text{int}})$. Then the critical strain intensity e_{int} necessary for plastic wrinkling of the considered annular plate (with inner radius a, outer radius 1 and thickness h is given as solution of the nonlinear equation

$$e_{\text{int}} = \overset{\circ}{e}_{\text{int}} + 0.408(1+3\alpha_0)\frac{h^2}{a^2} \tag{3.38}$$

where $\alpha_0 = \dfrac{E_K}{\frac{\Phi(e_{\text{int}})-\Phi(\overset{\circ}{e}_{\text{int}})}{e_{\text{int}}-\overset{\circ}{e}_{\text{int}}}}$. (E_K is the tangent modulus.)

3.5 Bifurcation equations with concentrated loads

Let us consider a thin plate (for simplicity a rectangular) which is loaded by a concentrated transverse pressure $\boldsymbol{p}$ in the point (x_0, y_0) of the middle surface as it is shown in Fig. 3.8 .

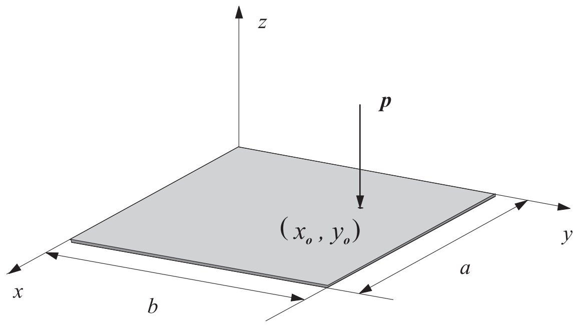

Figure 3.8: A rectangular plate with a concentrated load

For simplicity we assume the plate equation

$$D\triangle^2 w = -p(\boldsymbol{r} - \boldsymbol{r}_0) \tag{3.39}$$

with $p(\boldsymbol{r} - \boldsymbol{r}_0) = p_0\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ (δ the Dirac function, $p_0 > 0$ a parameter).

The potential energy of the plate in this case is

$$\Pi = \frac{D}{2}\iint_A (\triangle w)^2 dxdy \tag{3.40}$$

where $A = [0, a] \times [0, b]$. Let us seek the deflection of the plate as an eigenfunction expansion (see [36])

$$w(x, y) = \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} \tag{3.41}$$

with unknown coefficients a_{mn}. If we insert (3.41) into (3.40) we can calculate the associated potential energy by

$$\Pi = \frac{\pi^2 abD}{8} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} a_{mn} \left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2 . \tag{3.42}$$

The work of the concentrated load along the virtual displacement is given by

$$\delta W = -p_0 \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} a_{mn} \sin\frac{m\pi x_0}{a} \sin\frac{n\pi y_0}{b} . \tag{3.43}$$

Since this work is equal to the change of the potential energy we have an equation for determining the coefficients a_{mn} as (see [36])

$$a_{mn} = \frac{-p_0 4 \sin\dfrac{m\pi x_0}{a} \cdot \sin\dfrac{n\pi y_0}{b}}{\pi^4 abD \left[\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right]} \quad (m, n = 1, 3, \dots) . \tag{3.44}$$

Now the resulting solution w can be written as

$$w(x,y) = -\frac{4p_0}{\pi^2 Dab} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin\dfrac{m\pi x_0}{a} \cdot \sin\dfrac{n\pi y_0}{b}}{\left[\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right]^2} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{m\pi y}{b}\,. \tag{3.45}$$

It is clear that other geometrical situations (circular or annular discs with concentrated loads) can be investigated in the same way using the eigenfunctions of the circular resp. annular discs.

3.6 Plastic buckling and plastic flutter

In the metal spinning process instability may occur either by static buckling (neighboring equilibrium states exist) or by flutter or by both ways (at the same or at different loads). Even if the system is considered as statical, its instability behaviour may be dynamical. The results of the investigations [13, 23, 44, 48] indicate that multiple ranges of stability and instability may occur also in the presence of slight damping and that the variety of possibilities is even richer than in the absence of damping.

In certain cases it is possible to guarantee the static bifurcation route on the basis of the selfadjointness of differential operators occuring in the linearization of the problem.

Let us rewrite the homogenous perturbational system (2.69) in the more compact form with $\rho \equiv 1$ and the presence of inertia forces as

$$\boldsymbol{L}\boldsymbol{u} - \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \tag{3.46}$$

$$\boldsymbol{M}\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_F \times (0,T)\,, \tag{3.47}$$

$$\boldsymbol{N}\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_{\mathcal{F}} \times (0,T)\,, \tag{3.48}$$

where the definition of the linear operators follows immediately from (2.70) - (2.73).

In a parallel way we consider the associated static boundary value problem

$$\boldsymbol{L}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\,, \tag{3.49}$$

$$\boldsymbol{M}\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_F\,, \tag{3.50}$$

$$\boldsymbol{N}\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,. \tag{3.51}$$

We want to find vibrational solutions of (3.49) - (3.51) in the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x})e^{iwt} \tag{3.52}$$

where $\boldsymbol{U}$ is an unknown function of the phase variables and $\omega \in \mathbb{C}$ is an unknown frequency. This results in the following boundary value problems

$$\boldsymbol{L}\boldsymbol{U} + \omega^2 \boldsymbol{U} = 0 \quad \text{in} \quad \Omega\,, \tag{3.53}$$

$$\boldsymbol{M}\boldsymbol{U} = 0 \quad \text{on} \quad \Gamma_F\,, \tag{3.54}$$

$$\boldsymbol{N}\boldsymbol{U} = 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,. \tag{3.55}$$

It is well known ([5]) that a sufficient condition for the self-adjointness of (3.53) - (3.55) is the equality

$$\int_\Omega \left\{ \left[L\boldsymbol{U}^{(1)} + \omega^2 \boldsymbol{U}^{(1)} \right] \boldsymbol{U}^{(2)} - \left[L\boldsymbol{U}^{(2)} + \omega^2 \boldsymbol{U}^{(2)} \right] \boldsymbol{U}^{(1)} \right\} dV = 0\,, \tag{3.56}$$

which has to be satisfied for all perturbations $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}$ satisfying the boundary conditions (3.54), (3.55).

Suppose now that for a certain value ω system (3.49) has a nontrivial solution. If the system (3.53) is selfadjoint the associated number ω^2 is real. It turns out that in this case the loss of stability is statically, i.e., we have a buckling bifurcation in the variational system (2.69) - (2.73).

As an example let us discuss the plastic buckling behavior of a rectangular plate (Sect. 3.3). For this we consider the bifurcation equation (3.24) and derive conditions for plastic flutter (see also [43, 4]). To show this it is sufficient to investigate the plastic instability only in one phase variable direction, say in x-direction. Thus we have to consider the PDE problem

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} = 0\,. \tag{3.57}$$

Including inertia forces we come to the plastic wave equation (ρ denotes the material density)

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} = 0\,. \tag{3.58}$$

Let us assume the initial conditions

$$w(0,t) = w(a,t) = 0\ ,\ \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(a,t) = 0\ ,\ t > 0. \tag{3.59}$$

Now we try to find a plastic wave solution of (3.58), (3.59) in the form

$$w(x,t) = W(x)e^{i\omega t} \tag{3.60}$$

where $W(x)$ is an unknown function and $\omega \in \mathbb{C}$ is a parameter to be defined. Inserting (3.60) into (3.58), (3.59) we receive the ODE problem

$$(D_2 - D_3)W^{(4)} + \tau_{\text{int}} h W^{(2)} + \rho(-\omega^2)W = 0 \tag{3.61}$$

or

$$W^{(4)} + k^2 W^{(2)} - \widehat{\omega}^2 W = 0\,. \tag{3.62}$$

Here we have introduced the abbreviations

$$k = \sqrt{\frac{\tau_{\text{int}} h}{D_2 - D_3}} \text{ and } \widehat{\omega} = \omega\sqrt{\frac{\rho}{D_2 - D_3}}\,. \tag{3.63}$$

For the equation (3.62) it is well-known (see [5, 14]) that for small k there are two critical zeros of the determinant of the eigenvalue problem and both they are real. In this case the system is, with respect to the solution form (3.60), stable. For increasing k we have for a certain critical parameter k^* a double real zero of the determinant and for $k > k^*$ two critical complex zeros, one with positive and one with negative imaginary part. This shows that beginning with k^* the system becomes unstable. Now we can compute (using (3.63)) the critical intensity τ^*_{int} for dynamic plastic wrinkling as

$$\tau^*_{\text{int}} = k^{*^2} \frac{(D_2 - D_3)}{h} . \tag{3.64}$$

It is easy to see that the critical load (3.64) for dynamic wrinkling is smaller than the critical value for static buckling. As a consequence we conclude that the rectangular plate considered in Sect. 3.3 for small k shows static plastic buckling and for large k shows dynamic buckling. Thus the aspect ratio of the system is given by

$$a = \frac{\tau_{\text{int}} h}{D_2 - D_3} . \tag{3.65}$$

Let us note that the discussed properties of the eigenvalue problems of the bifurcation equation (3.57) are necessary conditions for a certain type of bifurcations in the full non-linear system. Let us assume that other sufficient conditions for nonlinear plastic buckling are satisfied (application of the center manifold theory or Lyapunov-Schmidt reduction process). Then the possible bifurcation scenario is the following. For small k the system undergoes a static plastic buckling bifurcation and for large k a plastic flutter (or Hopf) bifurcation.

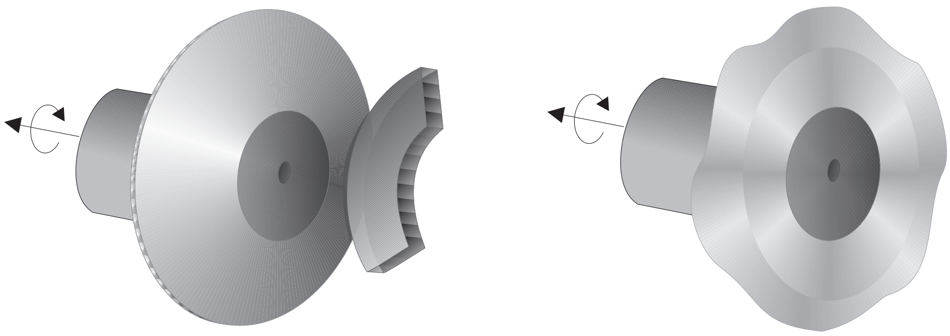

Figure 3.9: Plastic buckling (or flutter) in the spinning process

Theoretically it is possible that for a certain value of k there is one zero critical eigenvalue and two purely imaginary critical eigenvalues. In this situation the reduced bifurcation equation of the nonlinear system is of third order and the necessary conditions for (plastic) Ruelle-Takens-Newhouse bifurcations ([21, 45]) are satisfied. The bifurcation picture of the reduced ODE problem shows that heteroclinic orbits may occur ([21]) . A small perturbation leads to transversal heteroclinic points. Such points are a strong indication that chaotic motions might occur.

Other examples of coupling between divergence instability (in our case static buckling) and flutter are the fluid-carrying tube with an elastic support, the double pendulum and the tractor-semitrailor (he,tr,stz).

4 Large deformation shell theory in metal forming

4.1 Basic facts from large deformation shell theory

In order to describe the formation of wrinkles in the spinning process we need a large deformation theory for shells. In this section we give a short representation of such a theory which goes back to [14] (see also [9, 37, 50, 56]). Suppose that the middle surface of the shell (rotating blank) is given in parametric form

$$\boldsymbol{r} = \boldsymbol{r}(\xi^1, \xi^2) \tag{4.1}$$

with $(\xi^1, \xi^2) \in G \subset \mathbb{R}^2$. An arbitrary point near the middle surface is defined as

$$\boldsymbol{R}(\xi^1, \xi^2, \zeta) = \boldsymbol{r}(\xi^1, \xi^2) + \zeta \boldsymbol{n}(\xi^1, \xi^2) \tag{4.2}$$

where ζ is the distance between the point and the middle surface in the direction of the normal $\boldsymbol{n}$. In the following we need the vector functions

$$\boldsymbol{r}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \xi^\alpha} \qquad \text{and} \qquad \boldsymbol{n}_\alpha = \frac{\partial \boldsymbol{n}}{\partial \xi^i}\,. \tag{4.3}$$

Introduce also the first and the second fundamental form of the middle surface through

$$a_{\alpha\beta} = \boldsymbol{r}_\alpha \cdot \boldsymbol{r}_\beta \qquad \text{and} \qquad b_{\alpha\beta} = \boldsymbol{n} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial \xi^\alpha \partial \xi^\beta}\,. \tag{4.4}$$

The Christoffel symbols of the middle surface are defined by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} a^{\alpha\beta} \left[\frac{\partial a_{\rho\beta}}{\partial \xi^\gamma} + \frac{\partial a_{\gamma\rho}}{\partial \xi^\beta} - \frac{\partial a_{\beta\gamma}}{\partial \xi^\rho} \right]. \tag{4.5}$$

Suppose that Γ is a smooth curve on the middle surface. In any point of Γ we consider the unit vectors $\boldsymbol{n}$ (normal vector), $\boldsymbol{t}$ (tangent to Γ) and the vector $\boldsymbol{\nu} = \boldsymbol{t} \times \boldsymbol{n}$.

If s_t is the length parameter of the curve we introduce the values $t^\alpha = \dfrac{d\xi^\alpha}{ds_t}$ and define the numbers

$$\sigma_t = b_{\alpha\beta} t^\alpha t^\beta\,, \qquad \tau_t = -b_{\alpha\beta} t^\alpha t^\beta \tag{4.6}$$

and

$$\sigma_\nu = b_{\alpha\beta} \nu^\alpha \nu^\beta\,, \qquad \tau_\nu = -b_{\alpha\beta} \nu^\alpha \nu^\beta\,. \tag{4.7}$$

Assume that

$$\overset{\circ}{\boldsymbol{R}}(\xi^1,\xi^2,\zeta) = \overset{\circ}{\boldsymbol{r}}(\xi^1,\xi^2) + \zeta\overset{\circ}{\boldsymbol{n}}(\xi^1,\xi^2) \tag{4.8}$$

is the position of a point before deformation. (Here and in the following the sign $\circ$ denotes vectors or other values before deformation).

The same point after deformation has the position

$$\boldsymbol{R}(\xi^1,\xi^2,\zeta) = \boldsymbol{r}(\xi^1,\xi^2) + \lambda_\zeta(\xi^1,\xi^2)\left[\zeta + \frac{1}{2}\zeta^2\kappa_\zeta(\xi^1,\xi^2)\right]\cdot \boldsymbol{n}(\xi^1,\xi^2)\,, \tag{4.9}$$

where the functions λ_ζ and κ_ζ characterize the tension properties in the normal to the middle surface direction. As in small deformation theory (Section 3.2) we introduce with $\overset{\circ}{g} = \det(\overset{\circ}{g}_{ij})$ and $\overset{\circ}{a} = \det(\overset{\circ}{a}_{ij})$ the following integrals:

$$M^{\alpha\beta} = \lambda_\zeta \int_{-\overset{\circ}{h}/2}^{\overset{\circ}{h}/2} J\sigma^{\alpha\beta} d\zeta \quad , \quad \text{(moment tensor)} \tag{4.10}$$

$$Q^\alpha = \int_{-\overset{\circ}{h}/2}^{\overset{\circ}{h}/2} \left[\lambda_\zeta(1+\zeta\kappa_\zeta)J\sigma^{\alpha\beta} + \xi\left(\frac{\partial\lambda_\zeta}{\partial\xi^\beta}\right)J\sigma^{i\beta}\right]\cdot\sqrt{\overset{\circ}{g}/\overset{\circ}{a}}\,d\zeta \tag{4.11}$$

(transverse shear resultant) ,

and

$$N^{\alpha\beta} = \int_{-\overset{\circ}{h}/2}^{\overset{\circ}{h}/2} J\sigma^{\alpha\beta} d\zeta \tag{4.12}$$

(membrane forces tensor) .

In order to describe the equations of equilibrium we suppose that $\boldsymbol{q}\sqrt{a}\,d\zeta^1 d\zeta^2$ with $a = \det(a_{\alpha\beta})$ is the force intensity with respect to the unit surface element. The equilibrium position of the time-dependent thin shell is now described by the equations ([14])

$$\begin{aligned}
&\frac{\partial}{\partial\xi^\gamma}\left(\sqrt{\overset{\circ}{a}}N^{\gamma\alpha}\right) + \Gamma^\alpha_{\gamma\beta}\sqrt{\overset{\circ}{a}}\,N^{\gamma\beta} - b^\alpha_\gamma\sqrt{\overset{\circ}{a}}\,Q^\gamma + q^\alpha\sqrt{a} = 0\,,\\
&\frac{\partial}{\partial\xi^\alpha}\left(\sqrt{\overset{\circ}{a}}Q^\gamma\right) + b_{\gamma\beta}\sqrt{\overset{\circ}{a}}\,N^{\gamma\beta} + q_n\sqrt{a} = 0\,,\\
&\frac{\partial}{\partial\xi^\gamma}\left(\sqrt{\overset{\circ}{a}}\,M^{\gamma\alpha}\right) + \Gamma^\alpha_{\gamma\beta}\sqrt{\overset{\circ}{a}}\,M^{\gamma\beta} - \sqrt{\overset{\circ}{a}}\,Q^\alpha = 0\,.
\end{aligned} \tag{4.13}$$

If we exclude again as in Section 3.2 the shearing forces we get a reduced system in the form

$$\begin{aligned}
&\frac{\partial}{\partial\xi^\beta}\left[\sqrt{\overset{\circ}{a}}\,N^{\beta\alpha}\boldsymbol{r}_\alpha + \nabla_\gamma\left(\sqrt{\overset{\circ}{a}}\,M^{\alpha\beta}\right)\boldsymbol{n}\right] + \sqrt{a}\,\boldsymbol{q} = 0\,,\\
&\frac{\partial}{\partial\xi^\beta}\left[\sqrt{\overset{\circ}{a}}\,N^{\beta\alpha}\boldsymbol{r}_\alpha + \boldsymbol{r}_\beta\frac{\partial}{\partial\xi^\gamma}\left(\sqrt{\overset{\circ}{a}}\,M^{\gamma\mu}\boldsymbol{r}_\mu\right)\boldsymbol{n}\right] + \sqrt{a}\,\boldsymbol{q} = 0\,.
\end{aligned} \tag{4.14}$$

Note that we can also introduce special types of forces as realization of $\boldsymbol{q}$ such as

$$
\begin{array}{lll}
\text{normal forces} & \boldsymbol{F}_n = & q\boldsymbol{n}\,, \\
\text{inertia forces} & \boldsymbol{F}_{\mathcal{I}} = & -\overset{\circ}{h}\overset{\circ}{\rho}\, B^{-1/2}\, \frac{\partial^2 \boldsymbol{r}}{\partial t^2}\,, \\
\text{Coriolis forces} & \boldsymbol{F}_C = & -\omega\overset{\circ}{h}\overset{\circ}{\rho}\, B^{-1/2}\, \frac{\partial \boldsymbol{r}}{\partial t} \quad \text{and} \\
\text{centripetal forces} & \boldsymbol{F}_Z = & -\omega^2\overset{\circ}{h}\overset{\circ}{\rho}\, B^{-1/2}\, \boldsymbol{r}\,.
\end{array}
\tag{4.15}
$$

Let us now derive the constitutive law for the material. Suppose that $\Phi = \Phi\,(a_{\alpha\beta} + 2\zeta\kappa_{\alpha\beta}\,,\, \lambda_\xi^2 + 2\lambda_\xi^2\kappa_\xi)$ is the potential of energy. It follows that the Piola-Kirchhoff tensors may be calculated in a standard way by the formulas

$$
J\sigma^{\alpha\beta} = 2\frac{\partial \Phi}{\partial g_{\alpha\beta}} \quad \text{and} \quad J\sigma^{33} = 2\frac{\partial \Phi}{\partial g_{3J}}\,. \tag{4.16}
$$

Since we are interested in linear parts only we can write

$$
\begin{aligned}
J\sigma^{\alpha\beta} &= 2\Bigg\{\left(\frac{\partial \Phi}{\partial g_{\alpha\beta}}\right)_o + 2\zeta\left[\left(\frac{\partial^2 \Phi}{\partial g_{\gamma\delta}\partial g_{\alpha\beta}}\right)_o \kappa_{\gamma\delta} + \right. \\
&\qquad\qquad \left. + \left(\frac{\partial^2 \Phi}{\partial g_{33}\,\partial g_{\alpha\beta}}\right)_o \lambda_\zeta^2\kappa_\zeta\right]\Bigg\} \quad \text{and} \\
J\sigma^{33} &= 0\,.
\end{aligned}
\tag{4.17}
$$

Now we use the decomposition

$$
\begin{aligned}
J\sigma^{ij} &= (J\sigma^{ij})_{(o)} + \zeta(J\sigma^{ij})_{(1)} \qquad \text{with} \\
(J\sigma^{ij})_{(o)} &= 2\frac{\partial \Phi^o}{\partial a_{ij}} \quad , \quad \Phi^o = \Phi_{|\zeta=0}\,, \quad \text{and} \\
(J\sigma^{ij})_{(1)} &= 4\left[\frac{\partial^2 \Phi}{\partial a_{\alpha\beta}\,\partial a_{ij}}\kappa_{\alpha\beta} + \frac{\partial^2 \Phi^o}{\partial \lambda_\zeta^2 \partial a_{ij}}\lambda_\zeta^2\kappa_\zeta\right]\,.
\end{aligned}
\tag{4.18}
$$

Finally, the tangent modulus E^{ijkl} in the present situation can be written as

$$
E^{ijkl} = 2\left(\frac{\partial^2 \Phi^o}{\partial a_{ij}\,\partial a_{kl}} + \frac{\partial^2 \Phi^o}{\partial a_{ij}\partial a_{lk}}\right) - \tag{4.19}
$$

$$
-4\frac{\partial^2 \Phi^o}{\partial \lambda_\zeta^2\,\partial a_{ij}}\,\frac{\partial^2 \Phi^o}{\partial \lambda_\zeta^2\,\partial a_{kl}}\Bigg/\frac{\partial^2 \Phi^o}{(\partial \lambda_\zeta^2)^2}\,. \tag{4.20}
$$

Now we describe shortly the principle of virtual work. Let $\overset{\circ}{A}$ be a simply connected region on the undeformed middle surface defined by

$$
\overset{\circ}{A} = \left\{\boldsymbol{r}(\xi^1, \xi^2)\,,\, (\xi^1, \xi^2) \in \overset{\circ}{G}\right\}
$$

and enclosed by the curve

$$\overset{\circ}{C} = \left\{ \boldsymbol{r}(\xi^1, \xi^2)\,,\, (\xi^1, \xi^2) \in \partial \overset{\circ}{G} \right\} .$$

Suppose that $\overset{\circ}{\Gamma} = \partial \overset{\circ}{G}$ can be written as $\overset{\circ}{\Gamma} = \overset{\circ}{\Gamma}_u \cup \overset{\circ}{\Gamma}_\sigma$, where on $\overset{\circ}{\Gamma}_u$ the boundary conditions are given for $\boldsymbol{R}$ and on $\overset{\circ}{\Gamma}_\sigma$ the boundary conditions are given for the generalized Kirchhoff forces and the bending moment. Assume also that with respect to possible variations

$$\delta \boldsymbol{r} = 0 \quad \text{and} \quad \delta \boldsymbol{n} = 0 \quad \text{for all} \quad (\xi^1, \xi^2) \in \overset{\circ}{\Gamma}_u\,. \tag{4.21}$$

The version of the principle of virtual work which can be used for the Ritz method is then

$$\begin{aligned} \int_{\overset{\circ}{S}} \left[N^{\alpha\beta} \delta(a_{\alpha\beta}/2) + \frac{1}{\lambda_\zeta} M^{\alpha\beta} \delta \kappa_{\alpha\beta} \right] d\overset{\circ}{S} &= \\ = \int_{\overset{\circ}{S}} \sqrt{a}\, \boldsymbol{q} \cdot \delta \boldsymbol{r} d\xi^1 d\xi^2 &- \int_{\overset{\circ}{\Gamma}_\sigma} \left[N'_{\nu\nu} \boldsymbol{\nu} + N'_{\nu t} \boldsymbol{t} + N'_{\nu n} \boldsymbol{n} \right] \cdot d\boldsymbol{r} + \\ &+ \left[M_{\nu\nu} \boldsymbol{\nu} \cdot \delta \boldsymbol{n} \right] d\overset{\circ}{s}_t\,. \end{aligned} \tag{4.22}$$

If we use the representations

$$\boldsymbol{r}_i \cdot \delta \boldsymbol{r} = \delta u_i,\ \boldsymbol{n} \cdot \delta \boldsymbol{r} = \delta w,\ \boldsymbol{\nu} \cdot \delta \boldsymbol{r} = \delta u_\nu\ , \boldsymbol{t} \cdot \delta \boldsymbol{r} = \delta u_t, \tag{4.23}$$

$$\boldsymbol{\nu} \cdot \delta \boldsymbol{n} = \delta \vartheta_\nu,\ , \boldsymbol{r}_j \cdot \delta \boldsymbol{r}_i = \delta w_{ij} \quad \text{and} \quad \boldsymbol{r}_j \cdot \delta \boldsymbol{n}_i = \delta(\nabla_i \vartheta_j) \tag{4.24}$$

we get the scalar version of the principle as ([14])

$$\begin{aligned} \int_{\overset{\circ}{S}} \left[N^{\alpha\beta} \delta(a_{\alpha\beta}/2) + \frac{1}{\lambda_\zeta} M^{\alpha\beta} \delta \kappa_{\alpha\beta} \right] d\overset{\circ}{S} &= \\ = \int_{\overset{\circ}{S}} \left(q^\beta \delta u_\beta + q \delta w \right) d\overset{\circ}{S} &- \int_{\overset{\circ}{\Gamma}_\sigma} \left[N'_{\nu\nu} \delta u_\gamma + N'_{\nu t} \delta u_t + N'_{\nu\nu} \delta w + \right. \\ &\left. + M_{\nu\nu} \delta \vartheta_\nu \right] d\overset{\circ}{s}_t\,. \end{aligned} \tag{4.25}$$

4.2 Instability of a rotating shell and the speed of rotation

We consider a thin shell of revolution of constant thickness h. To describe the shell we introduce ([14, 49]) an orthogonal curvilinear coordinate system, connected with the meridians and parallels of a shell. A position of a point on the median surface of the shell is defined by an angle $\varphi\,(0 \le \varphi < 2\pi)$ and the length of the arc of generatrix $s\,(s_1 \le s \le s_2)$. The shell is limited with two parallels $s = s_1$ and $s = s_2$. The geometry is characterized by a function $B(s)$, which is the distance between the axis of symmetry and the median surface. We define also functions R_1 and R_2 which are the main radii of the curvature and a function $\Theta(s)$, which is the angle between the initial normal to the median surface and the axis of symmetry. These functions are expressed through B as

$$\frac{1}{R_1} = \frac{d\Theta}{ds} = -\frac{B''}{\sqrt{1-B'^2}}\,, \tag{4.26}$$

$$\frac{1}{R_2} = \frac{\sin\Theta}{B} = \frac{\sqrt{1-B^2}}{B}\,, \tag{4.27}$$

$$\cos\Theta = B',\ 0 \le \Theta < \pi\,. \tag{4.28}$$

At each point of the neutral surface we introduce a local system of cartesian coordinates, which are the axes of tangent to the meridian, the tangent to the parallel and the initial normal. The vector $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector in the local coordinate system.

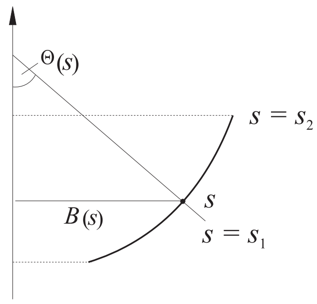

Figure 4.1: The orthogonal coordinate system

The shell rotates with the constant angular velocity vector ω. Suppose that $\boldsymbol{r}$ is the radius vector to a point on the neutral surface and $\boldsymbol{v}$ is the velocity of a shell point. Then $\boldsymbol{v} = \boldsymbol{\omega} \times (\boldsymbol{r}+\boldsymbol{u}) + \dot{\boldsymbol{u}}$ and the equation of motion is

$$\rho\ddot{\boldsymbol{u}} + \rho\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\boldsymbol{r}+\boldsymbol{u})] + 2\rho\boldsymbol{\omega} \times \dot{\boldsymbol{u}} + \nabla\Pi(\boldsymbol{u}) = 0 \tag{4.29}$$

where $\nabla\hat{\Pi} = \dfrac{\partial\hat{\Pi}}{\partial\boldsymbol{u}} - \dfrac{1}{B}\dfrac{\partial}{\partial s}\left(B\dfrac{\partial\hat{\Pi}}{\partial(\partial\boldsymbol{u}/\partial s)}\right) - \dfrac{\partial}{\partial\varphi}\dfrac{\partial\hat{\Pi}}{\partial(\partial\boldsymbol{u}/\partial\varphi)}$.

Here $\hat{\Pi}$ is the strain energy density, ρ is the density of the shell material.

We represent the displacement $\boldsymbol{u}(s,\varphi,t)$ as a sum of an axisymmetric displacement $\boldsymbol{u}^0(s)$ and the perturbational displacement $\boldsymbol{u}^1(s,\varphi,t)$, i.e.,

$$\boldsymbol{u}(s,\varphi,t) = \boldsymbol{u}^0(s) + \boldsymbol{u}^1(s,\varphi,t). \tag{4.30}$$

The expression for r and ω in local coordinates is

$$\boldsymbol{r} = (B\cos\Theta, 0, -B\sin\Theta), \quad \boldsymbol{\omega} = (-\hat{\omega}\sin\Theta, 0, -\hat{\omega}\cos\Theta). \tag{4.31}$$

Now we represent the strain energy density in the form

$$\hat{\Pi}(\boldsymbol{u}^1,\boldsymbol{u}^0) = \hat{\Pi}_1(\boldsymbol{u}^1) + \hat{\Pi}_f(\boldsymbol{u}^1,\boldsymbol{u}^0) + \hat{\Pi}_0(\boldsymbol{u}^0) + \hat{\Pi}_\omega(\boldsymbol{u}^1,\boldsymbol{u}^0) + \hat{\Pi}_*(\boldsymbol{u}^1,\boldsymbol{u}^0). \tag{4.32}$$

Here $\hat{\Pi}_1(\boldsymbol{u}^1)$ is a quadratic form of displacements, $\hat{\Pi}_f(\boldsymbol{u}^1, \boldsymbol{u}^0)$ is linear with respect to both $\boldsymbol{u}^1$ and $\boldsymbol{u}^0, \hat{\Pi}_0(\boldsymbol{u}^0)$ is a term depending only on $\boldsymbol{u}^0, \hat{\Pi}_\omega(\boldsymbol{u}^1, \boldsymbol{u}^0)$ is a quadratic form with respect to $\boldsymbol{u}^1$ and $\hat{\Pi}_*(\boldsymbol{u}^1, \boldsymbol{u}^0)$ contains all other terms. Further we will omit the index "^{n}" for displacements. Substituting (4.30) – (4.32) into equation (4.29) we get the equations

$$L_f(\boldsymbol{u}^0) = \boldsymbol{f}, \tag{4.33}$$

$$L(\boldsymbol{u}) + \omega^2 L_\omega(\boldsymbol{u}^0, \boldsymbol{u}) = \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + 2\,\omega\, L_C \frac{\partial \boldsymbol{u}}{\partial f} + \omega^2 L_e \boldsymbol{u} \tag{4.34}$$

with $\boldsymbol{f} = \omega^2 \boldsymbol{r},\ L_f(\boldsymbol{u}^0) = -\nabla \hat{\Pi}_f(\boldsymbol{u}^0, \boldsymbol{u})$,

$$L_C = \begin{bmatrix} 0 & \cos\Theta & 0 \\ -\cos\Theta & 0 & \sin\Theta \\ 0 & -\sin\Theta & 0 \end{bmatrix}, \quad L_e = \begin{bmatrix} -\cos^2\Theta & 0 & -\sin\Theta\cos\Theta \\ 0 & -1 & 0 \\ \sin\Theta\cos\Theta & 0 & -\sin^2\Theta \end{bmatrix},$$

$$L(\boldsymbol{u}) = -\nabla\hat{\Pi}(\boldsymbol{u}),\ L_f(\boldsymbol{u}^0) = -\nabla\hat{\Pi}_f(\boldsymbol{u}^0, \boldsymbol{u}),\ L_\omega(\boldsymbol{u}^0, \boldsymbol{u}) = -\frac{1}{\omega^2}\hat{\Pi}_\omega(\boldsymbol{u}^0, \boldsymbol{u}).$$

For linear and homogenous boundary conditions the solutions of (4.34) can be written as running waves in the form

$$u(s,\varphi,t) = \sum_{m=o}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=1}^{2} u_{mk}(s) C^l_{mk}\cos(m\varphi + \omega_{mk}t - \alpha^l)$$

$$v(s,\varphi,t) = \sum_{m=o}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=1}^{2} v_{mk}(s) C^l_{mk}\sin(m\varphi + \omega_{mk}t - \alpha^l)$$

$$w(s,\varphi,t) = \sum_{m=o}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=1}^{2} w_{mk}(s) C^l_{mk}\cos(m\varphi + \omega_{mk}t - \alpha^l)$$

In the paper [49], using the above representation of running waves, it is shown that for certain specific values of the speed of rotation ω, the critical eigenvalues of the linearization is zero and the shell becomes unstable. This effect of rotation is the most important on lower frequencies (see also [42]).

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