Electromagnetic wave propagation in complex materials with thermo-electric coupling

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1. Coupled system of Maxwell's equations and heat equation

Let $\Omega \subset \mathbb{R}^3$, $\partial\Omega$ the smooth enough boundary,

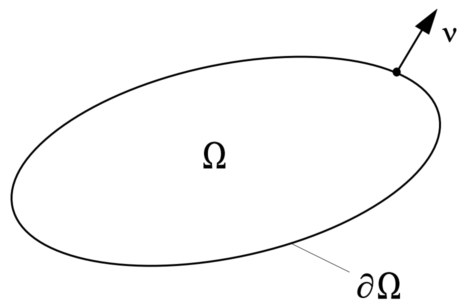


Fig. 1

$$\begin{aligned} &\varepsilon E_t + \sigma E = \mathrm{rot} H, \\ &\mu H_t + \mathrm{rot} E = 0, \\ &\theta_t - \Delta\theta = \sigma|E|^2, \end{aligned} \tag{1}$$

where $E(x,t)$ is the electric field, $H(x,t)$ is the magnetic field, $\theta(x,t)$ is the temperature, $\sigma(\theta)$ is the electrical conductivity, $\varepsilon(x)$ is the electric permittivity, $\mu(x)$ is the magnetic permeability, $(x,t) \in Q_T := \Omega \times [0,T)$.

$E(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t), x = (x_1, x_2, x_3),$

$$\mathrm{rotE} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} = \nabla \times \mathrm{E}$$

$q = \sigma|E|^2$ is Joule's heat

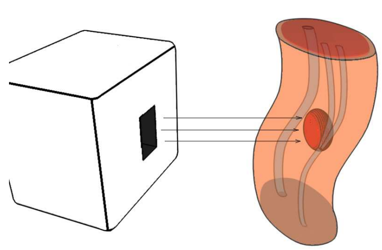


Fig. 2

Initial and boundary conditions
(Landau/Lifshetz, Electrodynamics):

$\nu \times E(x,t) = \nu \times G(x,t)$, where ν is the normal to the boundary $\partial\Omega$, $G(x,t)$ is some function.

$\theta_\nu = \nu \cdot \nabla\theta = 0$, $(x,t) \in S_T := \partial\Omega \times [0,T]$,

$E(x,0) = E_0(x), H(x,0) = H_0(x), \theta(x,0) = \theta_0(x), x \in \Omega,$

$E_0(x), H_0(x)$, and $\theta_0(x)$ are some functions.

2. The one-dimensional case

Suppose that $Q_T = \{(x,t)|0 < x < 1, 0 < t < T\}$ and $E(x,t) = (0, e(x,t), 0)$, $H(x,t) = (0,0,h(x,t))$.

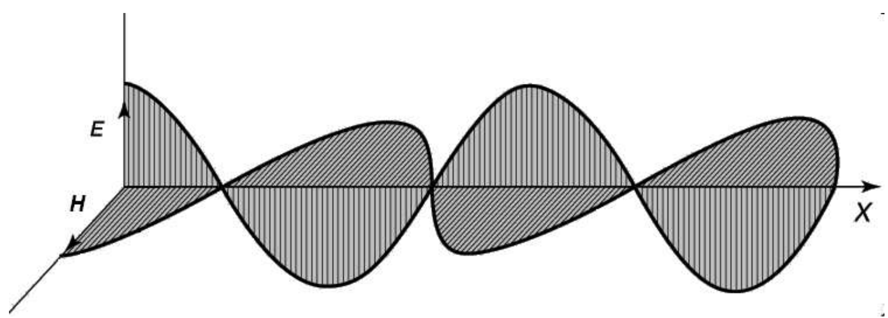


Fig. 3

Rewrite the previous system in the form

$$\begin{cases} \varepsilon e_t + \sigma e = -e_2 h_x\,, \\ \mu h_t + e_x = 0\,, \\ \theta_t - \theta_{xx} = \sigma|e|^2\,, \quad x \in (0,1), t > 0\,, \\ e(0,t) = f(t), \theta(0,t) = 0\,, \quad t > 0\,, \\ e(x,0) = e_0(x), h(x,0) = h_0(x)\,, \quad x \in (0,1)\,, \\ \theta(x,0) = \theta_0(x)\,. \end{cases} \tag{2}$$

We assume that $w(x,t) := \int\limits_0^t e(x,s)ds$ and $\varepsilon(x) = \mu(x) = 1$ and derive the following system:

$$\begin{cases} w_{tt} = w_{xx} - \sigma(\theta)w_t \,, \\ \theta_t - \theta_{xx} = \sigma(\theta)w_t^2 \,, \\ w(0,t) = f_1(t), w(1,t) = f_2(t) \,, \\ \theta(0,t) = \theta(1,t) = 0 \,, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) \,, \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{3}$$

Fig. 4

Assumptions

(A1) 1) $\sigma(z)$ is piecewise C^1 on $(0,\infty)$.

2) There exist constants $0 < \sigma_0 \leq \sigma_1$, such that $\sigma_0 \leq \sigma(z) \leq \sigma_1$, for any $z \geq 0$.

3) There exists a constant σ_2, such that $\sigma(z)z \leq \sigma_2$, for any $z \geq 0$.

(A2) 1) $\theta_0(x)$ is nonnegative and of class $C^2(0,1)$.

2) $w_0(x), w_1(x)$ are of class $C^4(0,1)$ and $f_1(t), f_2(t)$ are of class C^2.

Remark 1

Hölder spaces (Ladyzhenskaya, Solonnikov, Ural'ceva, 1968)

Suppose Ω is a bounded region in $\mathbb{R}^n$.

$$C^\alpha\left(\bar{\Omega}\right)=\left\{f\in C_b\left(\bar{\Omega}\right):\sup_{t,s\in\bar{\Omega},t\neq s}\frac{|f\left(t\right)-f\left(s\right)|}{|t-s|^\alpha}<+\infty\right\}$$

$$C^{\alpha,0}\left(\bar{\Omega}\times[a,b]\right)=\left\{f\in C\left(\bar{\Omega}\times[a,b]\right):f\left(\cdot,t\right)\in C^\alpha\left(\bar{\Omega}\right)\forall t\in[a,b]\right\}$$

$$C^{0,\alpha}\left(\bar{\Omega}\times[a,b]\right)=\left\{f\in C\left(\bar{\Omega}\times[a,b]\right):f\left(x,\cdot\right)\in C^\alpha[a,b]\,\forall x\in\bar{\Omega}\right\}$$

$$C^{\alpha,\frac{\alpha}{2}}\left(\bar{\Omega}\times[a,b]\right)=C^{0,\frac{\alpha}{2}}\left(\bar{\Omega}\times[a,b]\right)\cap C^{\alpha,0}\left(\bar{\Omega}\times[a,b]\right)$$

$$C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega}\times[a,b]\right)=\left\{f\in C^{2,1}\left(\bar{\Omega}\times[a,b]\right):\right.$$
$$\left.f_t,f_{x_ix_j}\in C^{\alpha,\frac{\alpha}{2}}\left(\bar{\Omega}\times[a,b]\right)\right\}.$$

Under the assumptions (**A1**)-(**A2**), the system has a classical unique global solution(Yin) (w,w_t,θ) on $Q_T=(0,1)\times(0,T)$, for any $T<\infty$. Furthermore, $w(x,t)\in C^{3,3}(\bar{Q}_T)$, $\theta(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$ for some $\alpha\in(0,1)$.

3. Almost periodic perturbations

Let $f_1(t)$ and $f_2(t)$ be are almost periodic functions.

Definition 1 A scalar continuous function $f:\mathbb{R}\to\mathbb{R}$, is called *almost-periodic in the sense of Bohr*, if for every $\varepsilon>0$ there exists an $L=L(\varepsilon)$ such that every interval $[t_0,t_0+L]$ contains at least one number α for which $|f(t+\alpha)-f(t)|<\varepsilon, t\in\mathbb{R}$.

Example 1

a) periodic functions: $\cos t,\sin t$ etc.

b) $a_1\cos\omega_1 t+a_2\cos\omega_2 t,\omega_1/\omega_2$ irrational, $a_1\neq 0,a_2\neq 0$

4. Cocycles in infinite-dimensional phase space

Definition 2 Let (P, ρ) be a complete metric space, $\{\tau^t\}_{t\in\mathbb{R}} : P \to P$ is a continuous map satisfying

1) $\tau^0(\cdot) = id_P$,

2) $\tau^{t+s}(\cdot) = \tau^t \circ \tau^s$, $\forall t, s \in \mathbb{R}$.

Then $(\{\tau^t\}_{t\in\mathbb{R}}, P)$ is called *base flow.*

Example 2

Shift map $\tau^t(p) := p + t, \ \forall t, p \in \mathbb{R} = P$

Definition 3 Let X be a complete metric space.
A *cocycle over the base flow* $\{\tau^t\}_{t\in\mathbb{R}}$ is a family of maps $\{\varphi^t(p, \cdot) : X \to X\}_{t\in\mathbb{R}, p\in P}$ such that

1) $\varphi^0(p, \cdot) = id_X, \forall p \in P$,

2) $\varphi^{t+s}(p, \cdot) = \varphi^t(\tau^s(p), \varphi^s(p, \cdot)), \forall t, s \in \mathbb{R}, \forall p \in P$.

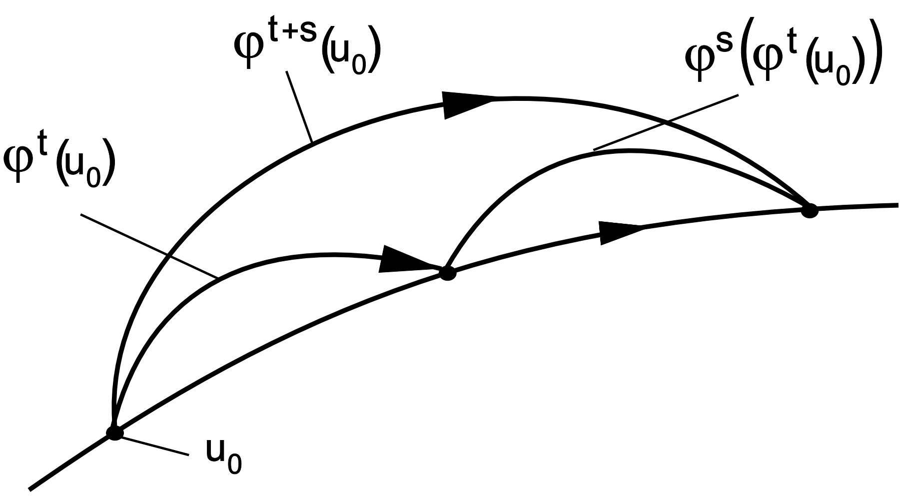


Fig. 5

5. Constructing a cocycle for the microwave heating problem

Introduce the auxiliary functions $f(x,t) = f_1(t)(x-1) + f_2(t)x$, $x \in (0,1), t \geq 0$, $W(x,t) = w(x,t) - f(x,t)$, $V(x,t) = W_t(x,t) - f_t(x,t)$. Rewrite equation (3) as the first order system

$$\begin{cases} W_t = V - f_t, \\ V_t = W_{xx} - \sigma(\theta)V + f_{tt}, \\ \theta_t - \theta_{xx} = \sigma(\theta)V^2, \\ W(0,t) = W(1,t) = 0, \theta(0,t) = \theta(1,t) = 0, \\ W(x,0) = w_0(x) - f(x,0), \\ W_t(x,0) = w_1(x) - f_t(x,0), \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{4}$$

Then (4) can be formally written as system

$$\frac{du}{dt} = Au + Bg(V,\theta) + F(t), \tag{5}$$

where $u = (W, V, \theta)$, $F(t) = (-f_t, f_{tt}, 0)$, A, B are linear operators.

If $(W(x,t), V(x,t), \theta(x,t))$ is a solution of (4), we can write it as

$$\begin{aligned} &u(t, t_0, u_0) = (W(\cdot,t), V(\cdot,t), \theta(\cdot,t)), \\ &u_0 = (w_0(x) - f(x,0), w_1(x) - f_t(x,0), \theta_0(x)). \end{aligned}$$

Consider the shift map $\tau^t : \mathbb{R}_+ \to \mathbb{R}_+$ and the family of mappings $\{\varphi^t(t_0,\cdot)\}_{t,t_0\in\mathbb{R}_+}$ such that

$$\begin{aligned} &\varphi^{(\cdot)}(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \times X \to X, \\ &\tau^t(t_0) = t + t_0, \\ &\varphi^t(t_0, u_0) = u(t + t_0, t_0, u_0), \end{aligned}$$

for any $t \in \mathbb{R}_+, t_0 \in \mathbb{R}_+, u_0 \in X$, where X is the Banach space $X = H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)$ with the norm $\| (W, V, \theta) \|_X^2 = \| W \|_{H_0^1}^2 + \| V \|_{L^2}^2 + \| \theta \|_{L^2}^2$.

6. Existence of an almost periodic solution

Definition 4 A continuous function $y : \mathbb{R}_+ \to X$ is called an *integral* of the cocycle on $\mathbb{R}_+$, if $\varphi^t(s, y(s)) = y(t+s)$ for all $t, s \in \mathbb{R}_+$.

The integral $y : \mathbb{R}_+ \to X$ of the cocycle is said to be

1) *uniformly stable* if for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ such that

$$\varphi^t(s, N_\delta(y(s))) \subset N_\varepsilon(y(t+s)),\ t, s \in \mathbb{R}_+\,,$$

2) *uniformly asymptotically stable* (UAS) if it is uniformly stable and there exist a $\delta_0 > 0$ with the property that for any $\varepsilon > 0$, there is a $t_0 > 0$ such that

$$\varphi^t(s, N_{\delta_0}(y(s))) \subset N_\varepsilon(y(t+s)),\ t \geq t_0, s \in \mathbb{R}_+\,.$$

Theorem 1 *System (2) has an almost periodic solution which is uniformly asymptotically stable.*

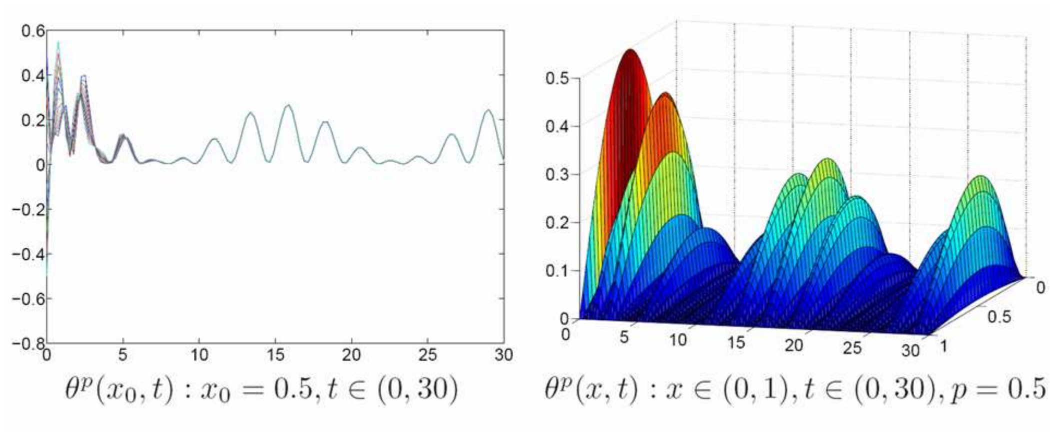
Idea of the proof: We consider the functional

$$\Phi(W, V, \theta) = \int_0^1 (W_x^2 + 2\lambda WV + V^2 + a\theta^2)dx,$$

where $\lambda > 0$ and $a > 0$ are some parameters and show that each solution of (4) is bounded in future, uniformly asymptotically stable and has a compact orbit.

Consider the system (4) with initial-boundary conditions

$$\begin{gathered} W(x,0) = p\sin(\pi x), \\ V(x,0) = 0, \theta(x,0) = 0, \\ W(0,t) = \sin(t) + \sin(\sqrt{2}t),\ W(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0. \end{gathered}$$



$\theta^p(x_0,t) : x_0 = 0.5, t \in (0,30)$ $\theta^p(x,t) : x \in (0,1), t \in (0,30), p = 0.5$

Fig. 6

Reference

[1] Yu. Kalinin, V. Reitmann and N. Yumaguzin, Asymptotic behavior of Maxwell's equation in one-space dimension with thermal effect, submitted for publication in AIMS Journals.

7. Existence of the global attractor for the 1-dimensional microwave heating problem

We consider the system

$$\begin{array}{ll} w_{tt} - w_{xx} + \sigma(\theta) w_t = 0, & 0 < x < 1, \quad t > 0 \\ \theta_t - \theta_{xx} = \sigma(\theta) w_t^2, & 0 < x < 1, \quad t > 0 \end{array} \tag{6}$$

with boundary conditions (defining the set Ξ)

$$\begin{array}{ll} w(0,t) = w(1,t) = 0, & t > 0 \\ \theta(0,t) = \theta(1,t) = 0, & t > 0 \end{array} \tag{7}$$

and initial conditions

$$w(x,0) = w_0(x), w_t(x,0) = w_1(x), \theta(x,0) = \theta_0(x), \quad x \in (0,1) \tag{8}$$

We make the following assumptions:

(A3) $w_0 \in C^3\,[0,1]\,, w_1 \in C^3\,[0,1]\,, \theta_0 \in C^{2+\alpha}\,[0,1]$

(A4) Compatibility conditions (defining the set Λ)

$\theta_0\,(0) = \theta_0\,(1) = 0$, $w_0\,(0) = w_0\,(1) = 0$,
$w_1\,(0) = w_1\,(1) = 0$

$\theta_0''\,(0) = \theta_0''\,(1) = 0$, $w_0''\,(0) = w_0''\,(1) = 0$

(A5) σ is piecewise C^1 on $[0,+\infty)$ and there exists $M > 0$ and $k \geq 0$ such that $0 \leq \sigma\,(z) \leq M(1+z)^k, \forall z \geq 0$

Theorem 2 *(Yin, 2001) Under the assumptions* **(A3) – (A5)** *the system (6) - (8) has a unique global solution* (w,θ) *with* $\theta \in C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{Q_T}\right)$*,* $w \in C^{3,3}\left(\bar{Q_T}\right)$ *for any* $T > 0$*, where* $Q_T = (0,1) \times (0,T]$*.*

Idea of the proof: Define the phase space

$$X = \left(C^3\,[0,1] \times C^2\,[0,1] \times C^{2+\alpha}\,[0,1]\right) \cap \Lambda \cap \Xi$$

The norm is induced from $H_0^1 \times L^2 \times H_0^1$

$\|(w,v,\theta)\|_X^2 = \|w_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta_x\|_{L^2}^2$, where

$\|u\|_{L^2} = (\int_0^1 u^2 dx)^{1/2}$.

Define the map $\varphi^t : X \to X$:

$$\varphi^t u_0 = u\,(t) = u\,(t, u_0)\,,$$

where $u\,(t) = (w\,(t)\,, v\,(t)\,, \theta\,(t))$ is the solution with initial data $u_0 = (w_0, w_1, \theta_0)$

$\sigma : \mathbb{R}_+ \to \mathbb{R}$ is called *Lipschitz* if $\exists L > 0\ \forall x, y \in \mathbb{R}_+$:

$|\sigma(x) - \sigma(y)| \leq L|x-y|$

Theorem 3 *Suppose additionally that σ is Lipschitz and $0 < \sigma_0 \leq \sigma(z) \leq \sigma_1, \forall z \geq 0$. Then the mapping $\varphi^t : X \to X$ is continuous for any $t \geq 0$.*

Proof $u = (w(t), v(t), \theta(t))$, $\bar{u} = \left(\bar{w}(t), \bar{v}(t), \bar{\theta}(t)\right)$ are solutions with initial data $u_0 = (w_0, v_0, \theta_0)$, $\bar{u}_0 = \left(\bar{w}_0, \bar{v}_0, \bar{\theta}_0\right)$.

$\tilde{w} := w - \bar{w}$

$\tilde{\theta} := \theta - \bar{\theta}$

$\tilde{w}_{tt} - \tilde{w}_{xx} + \left(\sigma(\theta) v - \sigma\left(\bar{\theta}\right) \bar{v}\right) = 0 \ \ (\cdot, \tilde{w}_t)$

$\tilde{\theta}_t - \tilde{\theta}_{xx} = \sigma(\theta) v^2 - \sigma\left(\bar{\theta}\right) \bar{v}^2 \ \ \left(\cdot, \tilde{\theta}_t\right)$

1)$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2\right) \leq C\left(\left\|\tilde{\theta}\right\|^2 + \|\tilde{w}_t\|^2\right)$

2) $\frac{1}{2}\frac{d}{dt}\left\|\tilde{\theta}_x\right\|^2 + \left\|\tilde{\theta}_t\right\|^2 \leq C\left\|\tilde{\theta}\right\|^2 + C\|\tilde{w}_t\|^2$

$\frac{d}{dt}\Phi(t) \leq C\Phi(t)$, and $\Phi(t) = \|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2 + \left\|\tilde{\theta}_x\right\|^2$

$\Phi(t) \leq \Phi(0) e^{ct}$

C denotes different constants that may depend on $T, \|u_0\|, \|\bar{u}_0\|$ ■

Theorem 4 *The initial-boundary problem (6) – (8) defines a dynamical system on X.*

8. Absorbing set

Definition 5 The set $A \subset X$ is called a *global attractor (B-attractor)* for the dynamical system $\left(\{\varphi^t\}, X\right)$ if it is compact, invariant and attracts all points (bounded sets) of the space X.

Definition 6 The set $Z \subset X$ is called *absorbing* for the dynamical system $\left(\{\varphi^t\}, X\right)$ if for any bounded that $M \subset X$ there exists a $t_0 = t_0(M)$ such that $\varphi^t M \subset Z, \forall t \geq 0$.

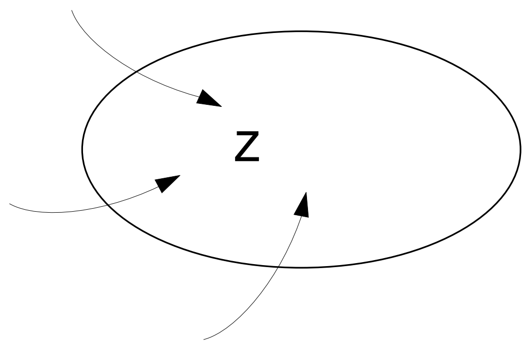


Fig. 7

Idea of the proof: Find a smooth functional Φ on X such that

$$\frac{d}{dt}\Phi\left(\varphi^t u\right) + c_1 \Phi\left(\varphi^t u\right) \leq D_1, \forall t > 0$$

$$\Phi(u) \geq c_2 \|u\|^2 - D_2, \forall u \in X$$

where $c_1, c_2, D_1, D_2 > 0$.

Additional assumptions

(A6) $\sigma(\theta)\theta \leq \sigma_2$ for $\theta < R$

(A7) $\exists \sigma_0, \sigma_1 0 < \sigma_0 \leq \sigma(\theta) \leq \sigma_1$

(A8) There exist $0 < \lambda < 1$ and $a > 0$ such that

$$4\lambda\sigma_0 - 4\lambda^2 - 2\lambda a \sigma_2 - \lambda^2 \sigma_1^2 > 0$$

$\|u\|_{L^\infty} = \operatorname{ess\,sup}\{|u(x)| : x \in (0,1)\}$

Theorem 5 *The solution components of (6) – (8) w_x and w_t decay to zero exponentially in $L^2(0,1)$, θ decays to zero in $L^\infty(0,1)$ with rate $Ct^{-\frac{1}{2}}$ and exponentially in $L^2(0,1)$, when $\|\theta\|_{L^\infty} < R$.*

$\|u(t)\|_{L^2} \leq C_1 e^{-C_2 t}\|u(0)\|, \forall t \geq 0$

Corollary 1 Each neighborhood of zero is an absorbing set for (6) – (8).

Idea of the proof:

Consider the functional on X

$$\Phi(w,v,\theta) = \int_0^1 \left(w_x^2 + 2\lambda wv + v^2 + a\theta^2\right) dx$$

Differentiating Φ with respect to the system we obtain (using the Poincare inequality and integration by parts):

$$\frac{d}{dt}\Phi(t) = \int_0^1 \left(-\lambda w_x^2 - \lambda\sigma(\theta) wv + \right.$$
$$\left.(\lambda - \sigma(\theta) + a\sigma(\theta)\theta) v^2 - a\theta_x^2\right) dx \leq -C_3\Phi$$

Theorem 6 *Let the assumptions 1), 2) be satisfied for $\|\theta\|_{L_2} > M$. Then the ball $\{(w,v,\theta : \Phi(w,v,\theta) < M\}$ is an absorbing set for the system (6) – (8). If additionally we have $\|\theta\|_{L_\infty} < R$, the attraction rate is exponential.*

9. Use of the frequency domain method

A similar system was considered by Likhtarnikov and Yakubovich (1977):

$$w_{tt} + 2\varepsilon w_t - \Delta w + \alpha w = f(\theta)\ ,$$
$$\theta_t - \beta\Delta\theta + w - \gamma g(\theta) = 0\,,$$

$$w(x,0) = w_0(x)\,,\ w_t(x,0) = w_1(x)\,,\ \theta(x,0) = \theta_0(x)\ .$$

Nonlinearities f and g satisfy the following conditions:

$$\begin{aligned}&\theta g(\theta) - f^2(\theta) \geq 0\,,\\ &\exists G : G(\theta) \geq 0 : G'(\theta) = g(\theta)\ .\end{aligned}$$

Consider the equation

$$\begin{aligned}\dot{y} &= Ay + B\xi\,,\\ w &= Cy\,.\end{aligned} \tag{9}$$

We introduce the spaces:

Y_0 is a Hilbert space

$Y_1 = D(A)$ is a Hilbert space dense in Y_0, A is a linear self-adjoint operator,

Y_{-1} is anti-dual to Y_1 (Y_{-1} is the closure of Y_0 with respect to the scalar product $(\eta, \varrho)_{-1} = \left(A^{-1}\eta, A^{-1}\varrho\right)_0$) ,

$Y_1 \subset Y_0 \subset Y_{-1}$ is a Hilbert space rigging structure,

Ξ and W are Hilbert spaces,

B, C are linear operators $\Xi \to Y_{-1}, Y_0 \to W$,

$F^j(y, \xi)$ are Hermitian forms on $Y_1 \times \Xi$, $j = 1, \ldots, m$,

$$F^j(y,\xi) = (F_1^j y, y)_{-1,1} + 2\,\mathrm{Re}\,(F_2^j y, \xi)_\Xi + (F_3^j\,\xi, \xi)_\Xi,$$

"Control": $\xi = \varphi(y)$

Consider the set $N = \cap_1^m N_j$ of processes $(y(\cdot), \xi(\cdot))$, satisfying the constraints

$$F^j(y(t), \xi(t)) \geq 0$$

(local constraint) or

$$\int_0^{t_k} F^j(y(t), \xi(t))\,dt \geq \gamma_j$$

(integral constraint)

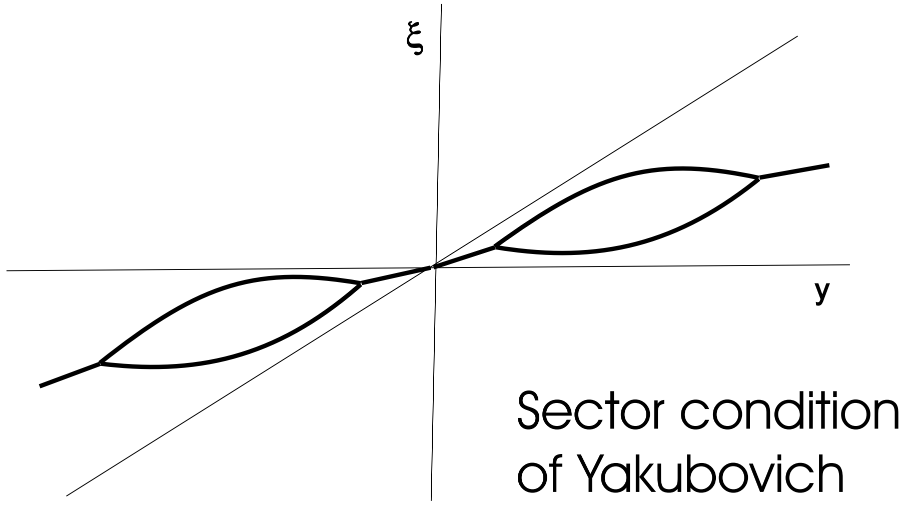


Fig. 8

$$(\Pi_0(i\omega)\xi,\xi) = \sum_{j=1}^{m} \tau_j F^j(\chi(i\omega)\xi,\xi) + \delta C\chi(i\omega)\|\xi\|^2,$$

where $\chi(i\omega) = (i\omega I - A)^{-1}B$ is the frequency characteristic of the system (9).

Definition 7 The system (9) is called *absolutely W-dichotomic* if for any process $p \in (L + p_0) \cap N$, the state of which is bounded $\|x(t)\|_0 \leq C_0 \quad \forall t \geq 0$, we have

$$w(\cdot) \in L^2(0,+\infty;W), \|w(\cdot)\|^2_{2,W} \leq C\left(\|x_0\|^2_0 + \sum \gamma_i\right). \tag{10}$$

Definition 8 The system (9) is called *absolutely W-stable*, if for any process $p \in (L + p_0) \cap N$ we have (10).

Theorem 7 *(The quadratic criterium of absolute stability of Likhtarnikov and Yakubovich)*

Let the following conditions hold:

1) The pair (A, B) is L^2-controllable ;

2) A has no spectral points in some neighborhood of the imaginary axis ;

3) The system (9) is minimally stable in the class N .

Then the system (9) is absolutely stable if and only if the frequency domain-condition is satisfied:

$$\exists \delta > 0 \quad \forall \omega \in \mathbb{R} \, \forall \xi \in \Xi \, :$$
$$F\left(\chi\left(i\omega\right)\xi, \xi\right) \leq -\delta \|\xi\|^2 \, .$$

10. Application of the frequency-domain method

Define the system

$$\dot{y} = Ay + B\xi \, ,$$

$$y(x,t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} w_t(x,t) \\ w(x,t) \\ \theta(x,t) \end{pmatrix} ,$$

$$A = \begin{pmatrix} -\sigma_0 I & -A_0 & 0 \\ I & 0 & 0 \\ 0 & 0 & -A_0 \end{pmatrix}, B = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}$$

A_0 is the operator defined by $-\Delta$ with homogeneous boundary conditions.

Notations:

$\sigma(\theta) = \sigma_0 + \hat{\sigma}(\theta), \sigma_0 > 0$

$$\xi(x,t) = \varphi\left(y\right) = \begin{pmatrix} -\sigma(\widehat{\theta}) w_t(x,t) \\ (\sigma_0 + \hat{\sigma}(\theta)) w_t^2(x,t) \end{pmatrix}$$

$\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of A_0,

$0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_k \to +\infty,$

$\{e_k\}_{k=1}^{\infty}$ are the eigenfunctions which define an orthonormized basis in L^2.

$w^k(t), \theta^k(t), \xi^k(t)$ are the Fourier coefficients of functions $w(x,t)$, $\theta(x,t)$,$\xi(x,t)$, respectively with respect to $\{e_k\}$.

$\tilde{u}(i\omega) = \int_0^\infty e^{-i\omega t} u(t) \mathrm{dt}$

Decompose solution components by the basis and take the Fourier transform in t $(k = 1, 2, ...)$:

$$-\omega^2 \tilde{w}^k(i\omega) + i\omega\sigma_0 \tilde{w}^k(i\omega) - \lambda_k \tilde{w}^k(i\omega) = \tilde{\xi}_1^k(i\omega)\,,$$
$$i\omega\tilde{\theta}^k(i\omega) + \lambda_k \tilde{\theta}^k(i\omega) = \tilde{\xi}_2^k(i\omega)\,.$$

From here we obtain:

$$\tilde{w}^k(i\omega) = \tilde{\xi}_1^k(i\omega)\chi_0(i\omega, \lambda_k)\,,$$
$$\tilde{\theta}^k(i\omega) = \tilde{\xi}_2^k(i\omega)\chi_1(i\omega, \lambda_k),$$

where

$$\chi_0(i\omega, \lambda_k) = \left(-\omega^2 + \sigma_0 i\omega + \lambda_k\right)^{-1},$$
$$\chi_1(i\omega, \lambda_k) = (i\omega + \lambda_k)^{-1},$$

$$\left(\Pi_0(i\omega)\,\tilde{\xi}, \tilde{\xi}\right) = \sum_k \left(\Pi_0^k(i\omega)\,\tilde{\xi}^k, \tilde{\xi}^k\right)\,.$$

Let us consider two quadratic constraints:

1) $F^1(y,\xi) = \int_0^1 y_1 \xi_1 dx = \int_0^1 w_t(x,t)\widehat{\sigma}(\theta) dx \geq 0\,,$

$$\left(\Pi_0^k(i\omega)\,\tilde{\xi}^k, \tilde{\xi}^k\right) = \mathrm{Re}\left(-\bar{\tilde{\xi}}_1^{\mathrm{k}} \tilde{\mathrm{w}}_{\mathrm{t}}^{\mathrm{k}}\right) = \frac{-\sigma_0\omega^2}{(\lambda_{\mathrm{k}} - \omega^2)^2 + \sigma_0^2\omega^2} \left|\bar{\tilde{\xi}}_1^{\mathrm{k}}\right|^2.$$

$$\Pi_0^k(i\omega) = \begin{pmatrix} \frac{-\sigma_0\omega^2}{(\lambda_k - \omega^2)^2 + \sigma_0^2\omega^2} & 0 \\ 0 & 0 \end{pmatrix}$$

Frequency-domain condition: $\Pi_0^k(i\omega) \leq 0, \ \forall \omega \in \mathbb{R}$ if $\sigma_0 > 0$.

Counterexample: blow-up if $\varphi_2(y) = \theta^3$

$\theta_t - \theta_{xx} = \theta^3$ on $(0,1)$ with Dirichlet BC.

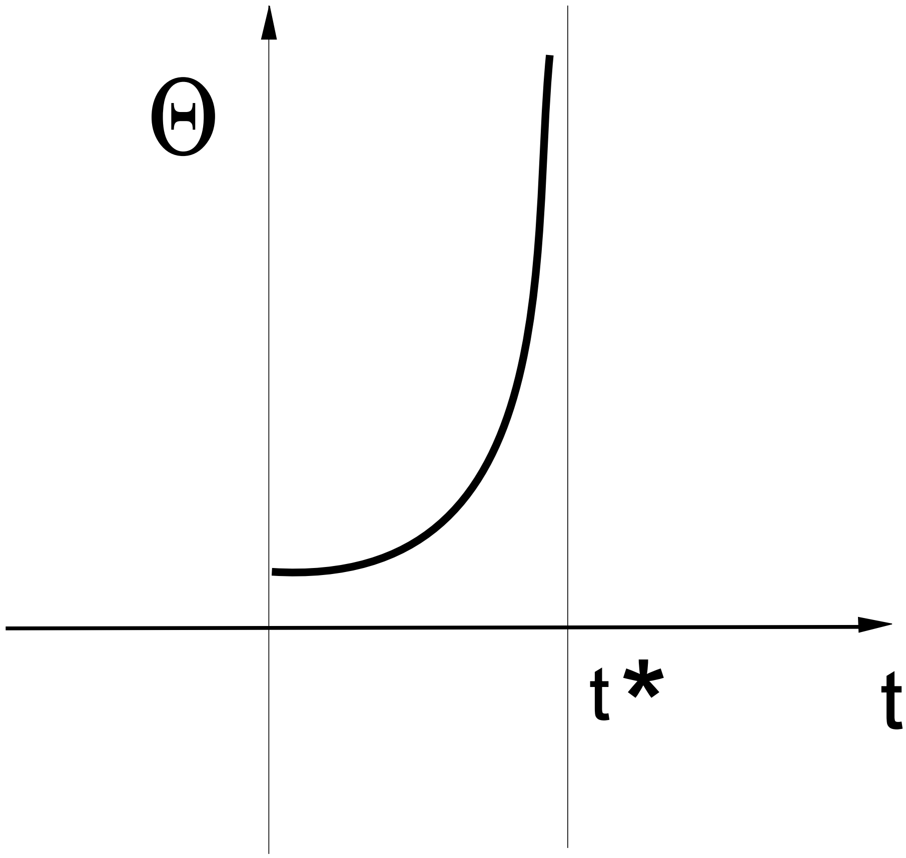


Fig. 9

2) $F^2(y,\xi) = \int_0^1 \left(\theta^2 - a\theta\xi_2\right) dx = \int_0^1 \left(\theta^2 - a\theta\sigma(\theta)\, w_t^2\right) dx$

Additional assumption :

$$\int_0^1 \left(\theta^2 - a\theta\sigma(\theta)\, w_t^2\right) dx > 0$$

$$F = F^1 + F^2$$

$$\Pi_0^k(i\omega) = \begin{pmatrix} \frac{-\sigma_0\omega^2}{(\lambda_k-\omega^2)^2+\sigma_0^2\omega^2} & 0\,, \\ 0 & \frac{1-a\lambda_k}{\omega^2+\lambda_k^2} \end{pmatrix}$$

$$a := \frac{2}{\lambda_1}$$

Frequency-domain condition: $\Pi_0^k(i\omega) \le 0,\ \forall \omega \in \mathbb{R}$.

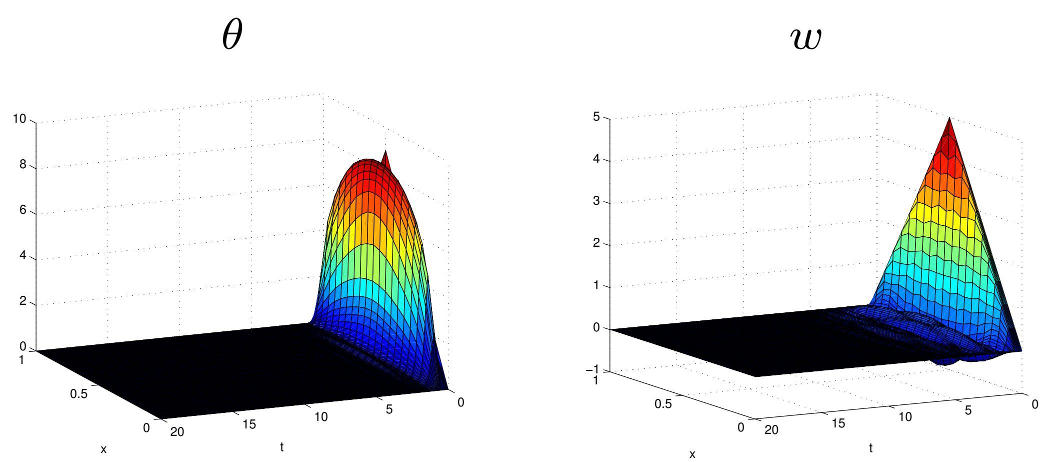
Theorem 7: Absolute stability with respect to the outputs w, w_t, θ.

Experimental results

$$1)\sigma(\theta) = \begin{cases} \exp(8-\theta), \theta > 4; \\ \exp(\theta), \theta \le 4 \end{cases}$$

$$\theta_0(x) = 9(1 - |2x-1|)$$

a) $w_0(x) = 5(1 - |2x-1|), w_1(x) = 0$



b) The same σ and θ_0,

$$w_0(x) = 0, w_1(x) = 5(1 - |2x-1|)$$

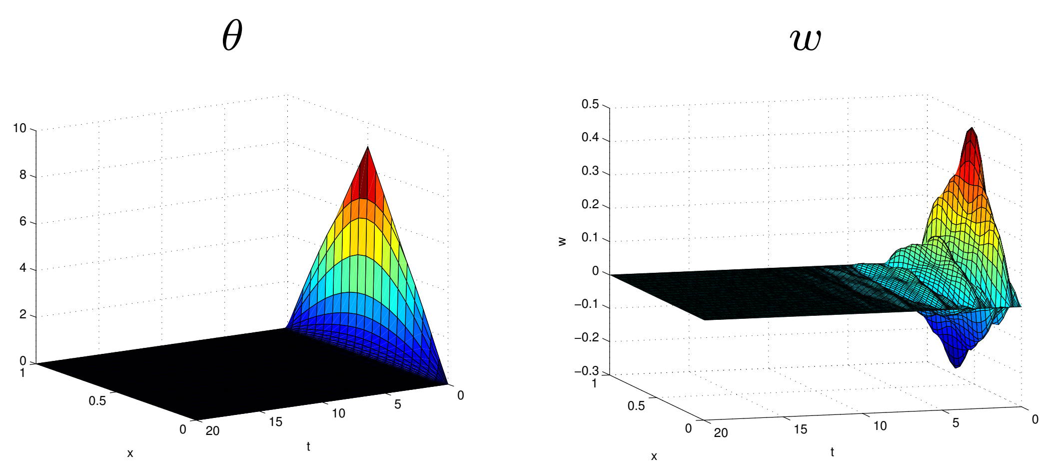


Fig. 10

2) $\sigma(\theta) = \begin{cases} \theta - 3, \theta > 4; \\ 5 - \theta, \theta \leq 4 \end{cases}$

$\theta_0(x) = 9(1 - |2x - 1|)$

$w_0(x) = 5(1 - |2x - 1|)$

$w_1(x) = 0$

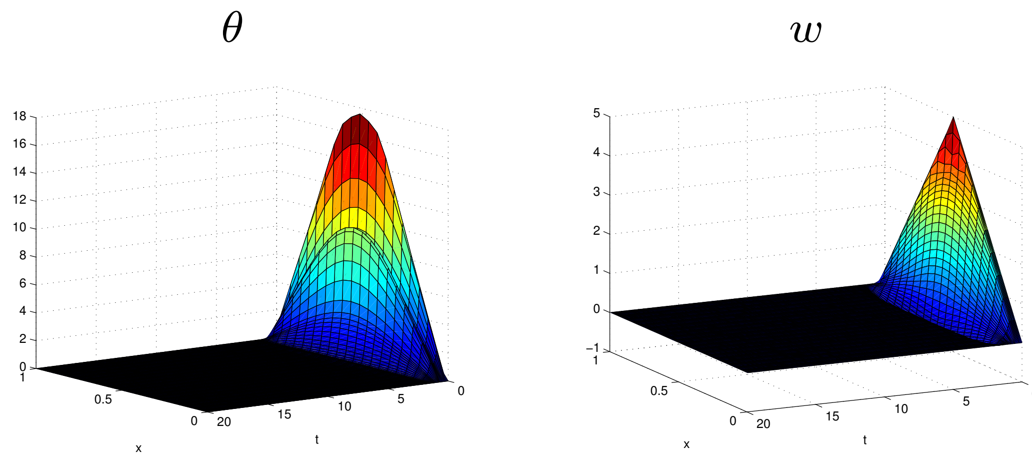


Fig. 11

11. Absolute stability in the microwave heating problem

Let us consider the following system:

$$\begin{aligned}
&w_t = v, && x \in (0,1),\ t > 0, \quad (11)\\
&v_t = w_{xx} - \sigma(\theta)v, && x \in (0,1),\ t > 0, \quad (12)\\
&k(\theta)_t = \theta_{xx} + \sigma(\theta)v^2, && x \in (0,1),\ t > 0, \quad (13)\\
&w(0,t) = 0,\ w(1,t) = 0, && t > 0, \quad (14)\\
&\theta(0,t) = \theta(1,t) = 0, && t > 0, \quad (15)\\
&w(x,0) = w_0(x),\ v(x,0) = w_1(x), && x \in (0,1), \quad (16)\\
&\theta(x,0) = \theta_0(x), && x \in (0,1). \quad (17)
\end{aligned}$$

In equation (13), $k(\theta)$ is a function defining internal energy of the material at the given temperature θ. With respect to Joule's law and heat capacity definition we have

$$k(\theta)_t = \frac{\partial k}{\partial \theta}\theta_t = c(\theta)\theta_t,$$

where c is the heat capacity of the material.

Thus, equation (13) takes the classic form of the heat transfer equation

$$c\theta_t = \theta_{xx} + F.$$

Assumptions

(A9) 1) $\sigma(z)$ is piecewise C^1 on $(0, \infty)$, $k(z)$ is a smooth function on $(0, \infty)$.
2) There exist $\sigma_0,\ \sigma_1$: $0 < \sigma_0 \leq \sigma_1$, such that $\sigma_0 \leq \sigma(z) \leq \sigma_1$, for $z \geq 0$.
3) There exists $k_1 > 0$, such that $k_1 \leq k'(z)$, for $z \geq 0$.
4) Either exists σ_2, such that $\sigma(z)z \leq \sigma_2$, or k_2, such that $k'(z)^{-1}z \leq k_2$, for $z \geq 0$.

(A10) 1) $\theta_0(x)$ is nonnegative and of class $C^2(0, 1)$. Compatibility conditions up to the second order hold at the points $(0, 0)$ and $(1, 0)$.
2) $w_0(x)$ and $w_1(x)$ are of class $C^4(0, 1)$. Compatibility holds at $(0, 0)$ and $(1, 0)$.

(Yin): Under conditions **(A9) – (A10)**, the system (11)-(17) has a unique classical solution $u = (w, v, \theta)$ on $Q_T = (0, 1) \times (0, T)$, for $T < \infty$. Furthermore, $w(x, t) \in C^{3,3}(\bar{Q}_T),\ \theta(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}_T)$ for some $\alpha \in (0, 1)$.

Consider the family of mappings $\{\varphi^t(\cdot)\}_{t \in \mathbb{R}_+}$, $\varphi^{(\cdot)}(\cdot) : \mathbb{R}_+ \times X \to X$ of the form

$$\varphi^t(u_0) = u(t, u_0)\,,$$

where $u(t, u_0) = (w(t), v(t), \theta(t))$ is the solution of the system (11)-(17) with initial conditions $u_0 = (w_0, w_1, \theta_0)$, and X is the associated solution space.

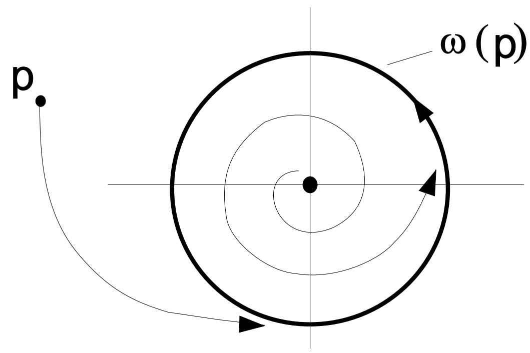


Fig. 12

Proposition 1 *Let φ be a continuous semigroup on a metric space X. If the trajectory through $x \in X$, $\gamma(x) := \cup_{t>0}\varphi^t(x)$ is precompact in X, then $\varphi^t(x) \to \omega(x),\ t \to \infty$. Here $\omega(x)$ is the*

ω-limit set of x, which is a compact subset of X. Moreover, $\omega(x)$ is positively invariant with respect to φ^t.

Definition 9 A *Lyapunov functional* for φ^t is a mapping $\Phi : X \to \mathbb{R}$, such that $\Phi(\varphi^t(x)) \leq \Phi(x)$, for any $x \in X,\ t \geq 0$.

Theorem 8 *(Dafermos) Let the conditions of Proposition1 hold, and let Φ be a Lyapunov functional for φ. Then Φ is constant on $\omega(x)$.*

12 Use of the invariance principle for the microwave heating problem

Introduce the Banach space $X = H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)$, with the norm

$$\|(w, v, \theta)\|_X^2 = \|w\|_{H_0^1}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2.$$

For any $(w, v, \theta) \in X$ we introduce the functional

$$\Phi(w, v, \theta) = \int_0^1 (w_x^2 + 2\lambda w v + v^2 + a\theta^2)dx, \qquad (18)$$

where λ and a are some positive constants.

(A11) There exist positive parameters λ and a such that for some $\epsilon > 0$

1) $0 < \lambda < 1$;

2) $\frac{1}{2}\sigma_0\epsilon - 1 < 0$;

3) $\lambda(\frac{1}{2\epsilon}\sigma_1 + 1) + ak_1^{-1}\sigma_2 - \sigma_0 < 0$ (σ_0, σ_1, σ_2 and k_1 from the conditions **(A9)**).

Lemma 1 *Under the conditions **(A9) – (A11)** the functional Φ of the system (11)-(17) has the following properties:*

1) There exist constants $C_1, C_2 > 0$, such that

$$C_1(\|w_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2) \leq \Phi(w, v, \theta) \qquad (19)$$
$$\leq C_2(\|w_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2), \qquad (20)$$

for $(w, v, \theta) \in X$.

2) There exists a constant $C_3 > 0$, such that

$$\frac{d}{dt}\Phi(t) \leq -C_3\Phi(t) \leq 0, \quad \textit{for all } t \geq t_0, \qquad (21)$$

where $\Phi(t) = \Phi(w(t), v(t), \theta(t))$ and $(w(t), v(t), \theta(t))$ is some solution of (11)-(17).

Idea of the proof: 1) This inequality follows from the estimate $\|w\|_{L^2}^2 \leq \|w_x\|_{L^2}^2$ for functions from $H_0^1(0, 1)$.

2) Direct computation shows that

$$\frac{d}{dt}\Phi(t) = 2\int\limits_0^1 \Big(-\lambda w_x^2 - \lambda\sigma(\theta)wv + (\lambda - \sigma(\theta) +$$

$$ak'(\theta)^{-1}\sigma(\theta)\theta)v^2 - ak'(\theta)^{-1}\theta_x^2\Big)dx \leq$$

$$-C_3\int\limits_0^1 \Big(\|w_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2\Big)dx,$$

where $C_3 > 0$ is a constant that exists due to **(A11):**

1) $\frac{1}{2}\sigma_0\epsilon - 1 < 0$;

2) $\lambda(\frac{1}{2\epsilon}\sigma_1 + 1) + ak_1^{-1}\sigma_2 - \sigma_0 < 0$ or
$\lambda(\frac{1}{2\epsilon}\sigma_1 + 1) + a\sigma_2\sigma_1 - \sigma_0 < 0$.

Theorem 9 *If the conditions of the Lemma hold, then any solution of the system (11)-(17) tends in X to the zero solution as $t \to \infty$.*

Proof Define Φ as $\Phi(t) = \Phi(w(t), v(t), \theta(t))$, where $(w(t), v(t), \theta(t))$ is some solution of (11)-(17).

Statement 2) of the lemma:

$$\frac{d}{dt}\Phi(t) \leq -C_3\Phi(t), \quad t \geq t_0. \tag{22}$$

Continuity of the functional and Gronwall's inequality:
$\lim\limits_{t\to\infty} \Phi(t) = 0$.

13 Numerical experiments

Initial conditions: $w_0(x) = 0$, $w_1(x) = p\sin(\pi x)$ and $\theta_0(x) = p\sin(\pi x)$ for solution components $w^p(x,t)$ and $\theta^p(x,t)$, where p is taken from $[-0.5, 0.5]$.

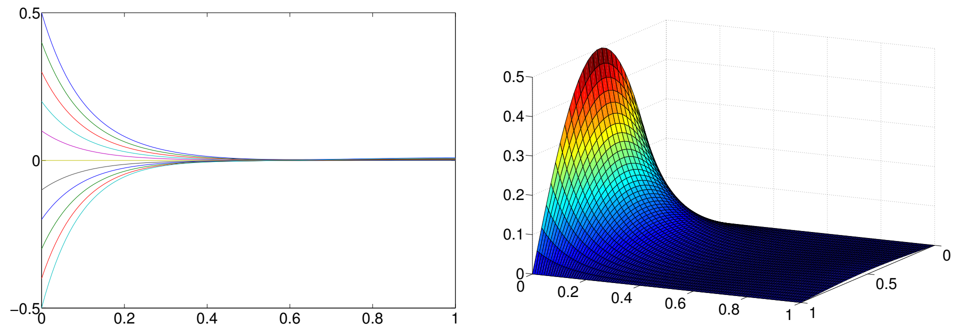


Fig. 13 Solution component $\theta^p(x,t)$.

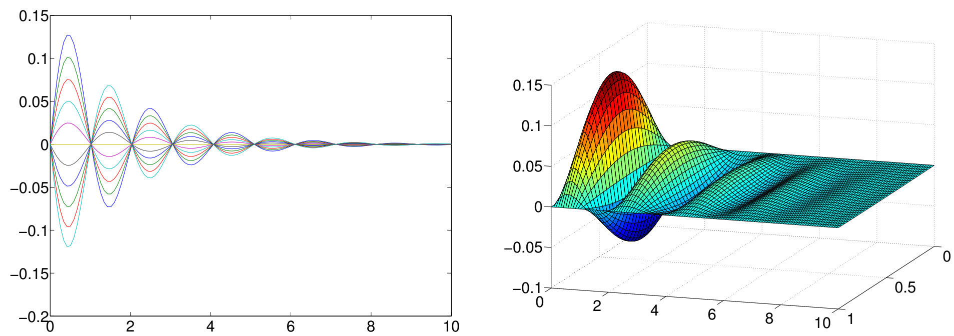


Fig. 14 Solution component $w^p(x,t)$.

14 The two-phase problem

We consider a coefficient of the form

$$k(s) = \begin{cases} s + 1, & s > \theta_c \\ s, & s \leq \theta_c \end{cases},$$

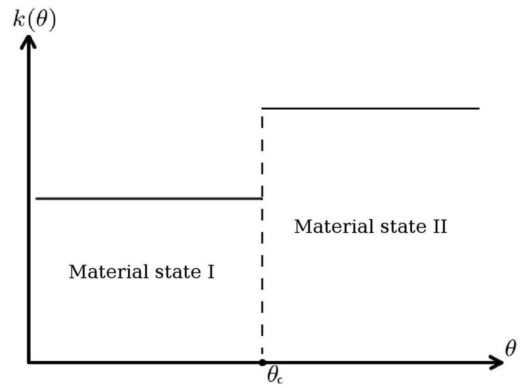


Fig. 15 Noncontinuous two-phase coefficient.

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